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SOLUTIONS OF THE EXAMPLES

IN

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SOLUTIONS OF THE EXAMPLES

IN

THE ELEMENTS OF HYDROSTATICS

BY

S. L. LONEY, M.A.

PROFESSOR AT THE ROYAL HOLLOWAY COLLEGE,

SOMETIME FELLOW OF SIDNEY SUSSEX COLLEGE, CAMBRIDGE.

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PREFACE.

THE following Solutions form a companion volume to the Solutions of the questions in my *Elements of Statics and Dynamics.* I hope they will be found useful to Teachers and Private Students.

S. L. LONEY.

ROYAL HOLLOWAY COLLEGE,
EGHAM, SURREY.
Nov. 4, 1902.

ELEMENTS OF HYDROSTATICS.

SOLUTIONS.

EXAMPLES. I. (Page 12.)

1. $$\frac{\text{wt.}}{1\text{ kilog. wt.}}=\frac{\text{area of large piston}}{\text{area of small piston}}=\frac{25^2}{2^2}=\frac{625}{4}.$$

$$\therefore\ \text{wt.}=156{\cdot}25\text{ kilos.}$$

2. $$\frac{\text{Force}}{1\text{ ton wt.}}=\frac{\frac{1}{4}}{100}=\frac{1}{400}.$$

$$\therefore\ \text{Force}=\frac{2240}{400}=5{\cdot}6\text{ lbs. wt.}$$

3. If W be the weight, then

$$W\times 4\times 144=1500.$$

$$\therefore\ W=\frac{1500}{576}=2\tfrac{29}{48}\text{ lbs.}$$

4. Let P be the thrust applied to the smaller piston, so that

$$\frac{P}{1\text{ ton wt.}}=\frac{1}{8^2},\ \textit{i.e.}\ P=35\text{ lbs. wt.}$$

$$\therefore\ \text{ratio of arms}=\frac{35}{5}=7:1.$$

5. If P be the thrust on the smaller piston and x on the other end of the lever, then

$$\frac{P}{10\text{ tons' wt.}}=\frac{3^2}{72^2}=\frac{1}{576},\ \text{and}\ x\times 24=P\times 2.$$

$$\therefore\ x=\frac{P}{12}=\frac{10\times 2240}{576\times 12}=3\tfrac{13}{54}\text{ lbs. wt.}$$

The greatest weight

$$=150\times\pi\,.\,72^2\text{ lbs.}=\frac{22}{7}\times\frac{150\times 5184}{2240}\text{ tons' wt.}=1091\tfrac{1}{49}\text{ tons' wt.}$$

6. The blow which is given to the cork is communicated also to each element of the surface of the vessel which is of the same size as the cork. Hence there is a very large amount of force applied over the surface of the vessel and thus it may be broken.

7. If p be the required pressure, then the thrust on the valve is p. Hence

$$2 \, . \, p = 24 \, . \, 12.$$

$$\therefore \; p = 144 \text{ lbs. wt. per sq. inch.}$$

8.
$$p \times \pi \left(\frac{1}{16}\right)^2 = \frac{1}{2}.$$

$$\therefore \; p = \frac{128}{\pi} \text{ lbs. wt. per sq. inch} = 40\tfrac{8}{11} \text{ lbs. wt. per sq. inch.}$$

9.
$$p \times \frac{1}{5} = 16.$$

$$\therefore \; p = 80 \text{ lbs. wt. per sq. inch.}$$

EXAMPLES. II. (Pages 17, 18.)

1. $9 \times 62\frac{1}{2} = 562\frac{1}{2}$ lbs. wt.
2. 8000 ozs. per cub. ft. $= 8000 \div 1728$ ozs. per cub. in.
3. $10 \times 13{\cdot}598 = 135{\cdot}98$ lbs. wt.
4. $13{\cdot}6 \times 1000$ grammes' wt.
5. $96\frac{1}{4} \times s = 13 \times 19{\cdot}25, \quad \therefore \; s = \frac{13}{5} = 2{\cdot}6.$
6. 2240 lbs. $= x \times 19\frac{1}{4} \times 62\frac{1}{2}$.

$$\therefore \; x = \frac{2240 \times 4 \times 2}{77 \times 125} = 1\tfrac{237}{275} \text{ cub. ft.}$$

7. Let V cub. cms. be the volume of a kilo. of cast copper, and V' that of the same wt. of copper wire, so that

$$V \times 8{\cdot}88 = 1000 = V' \times 8{\cdot}79.$$

$$\therefore \; V' - V = 1000 \left[\frac{1}{8{\cdot}79} - \frac{1}{8{\cdot}88}\right] = \frac{90}{8{\cdot}79 \times 8{\cdot}88} = 1{\cdot}153 \ldots \text{ cub. cms.}$$

8. Area of the section in sq. ft.

$$= \pi \left[\left(\frac{7}{2}\right)^2 - 2^2\right] \div 144 = \frac{11\pi}{192}.$$

$$\therefore \; 64{\cdot}4 = \frac{11\pi}{192} \times 1 \times s \times 62\tfrac{1}{2}.$$

$$\therefore \; s = \frac{644}{10} \times \frac{192 \times 7}{11 \times 22} \times \frac{2}{125} = 5{\cdot}72 \ldots .$$

9. $3 = 1\tfrac{1}{2} \times a \times 8{\cdot}8 \times 1000$, where a is the area in sq. ft.

$$\therefore \; a = \frac{2}{8800} \text{ sq. ft.} = \frac{9}{275} \text{ sq. in.}$$

10. $$2376 \times 10^3 = \frac{4}{3}\pi \times 45^3 \,.\, \rho = \frac{88}{21} \times 45^3 \times \rho.$$

$$\therefore \; \rho = \frac{2376 \times 21}{11 \times 9^3} = \frac{56}{9} = 6\tfrac{2}{9}.$$

11. $139{\cdot}0625 \times 10^3 = V \times 8{\cdot}9$, where V is the volume in cub. cms.

$$\therefore \; V = \frac{1390625}{89} = 15625 \text{ cub. cms.} = {\cdot}015625 \text{ cub. metre.}$$

12. The density $= \dfrac{4900}{9}$ lbs. per cub. ft.

$= \dfrac{4900}{9} \times {\cdot}01602$ grammes per cub. cm. nearly (Art. 17)

$= 8{\cdot}722$ grammes per cub. cm.

13. The density $= \dfrac{36000 \times 10^3}{45 \times 10^6}$ grammes per cub. cm.

$= \dfrac{36}{45} \times 62{\cdot}4$ lbs. per cub. ft. nearly (Art. 17)

$= 49{\cdot}9\ldots$ lbs. per cub. ft.

14. V = vol. of the casting; V' = proper volume, so that

$$V \times 6{\cdot}3 = V' \times 7{\cdot}5.$$

$$\therefore \; \frac{V - V'}{V} = 1 - \frac{V'}{V} = 1 - \frac{63}{75} = \frac{12}{75} = \frac{16}{100}. \quad \therefore \text{ etc.}$$

EXAMPLES. III. (Pages 21, 22.)

1. Here $\quad {\cdot}85 = \dfrac{V_1 \times 1 + V_2 \times {\cdot}8}{V_1 + V_2}. \quad \therefore \; V_1 : V_2 :: 1 : 3.$

2. By Art. 25,

$$s = \frac{12 + 20}{\dfrac{12}{1{\cdot}1} + \dfrac{20}{{\cdot}9}} = \frac{32 \times 1{\cdot}1 \times {\cdot}9}{10{\cdot}8 + 22} = \frac{31{\cdot}68}{32{\cdot}8} = {\cdot}9658\ldots.$$

3. $$\bar{\rho} = \frac{39 \times {\cdot}9 + 51 \times {\cdot}75}{39 + 51} = \frac{73{\cdot}35}{90} = {\cdot}815.$$

4. Here
$$1{\cdot}05=\frac{W_1+27}{\frac{W_1}{1}+\frac{27}{1{\cdot}08}}.$$
$$\therefore\ W_1\times{\cdot}05=27-27\times\frac{1{\cdot}05}{1{\cdot}08}=27\times\frac{3}{108}=\frac{3}{4}.$$
$$\therefore\ W_1=15\text{ ozs.}$$

5. Here
$$1=\frac{W_1+500}{\frac{W_1}{{\cdot}5}+\frac{500}{7}}.$$
$$\therefore\ W_1=500-\frac{500}{7}=\frac{3000}{7}\text{ ozs.}$$

If V_1 be the volume, then
$$\frac{3000}{7}=V_1\times{\cdot}5\times1000.\quad\text{Hence } V_1=\frac{6}{7}\text{ cub. ft.}$$

6. Let V_1 and V_2 be the respective volumes. Then
$$V_1+V_2=452,$$
and
$$7V_1+8{\cdot}9V_2=3373.$$
Hence solving $\quad V_1=342\ \text{ and }\ V_2=110.$

7. After B is filled from A the density of the liquid in it
$$=\frac{\rho_1+\rho_2}{2}.$$

After C is filled from B the density of the liquid in C
$$=\frac{1}{2}\left[\frac{\rho_1+\rho_2}{2}+\rho_3\right]=\frac{\rho_1+\rho_2+2\rho_3}{4}.$$

8.
$$4=\frac{V_1s_1+V_1s_2}{V_1+V_1}=\frac{s_1+s_2}{2},$$
and
$$3=\frac{W_1+W_1}{\frac{W_1}{s_1}+\frac{W_1}{s_2}}=\frac{2s_1s_2}{s_1+s_2}=\frac{s_1s_2}{4}.$$
$$\therefore\ s_1+s_2=8,\quad s_1s_2=12.$$
$$\therefore\ s_1=6,\quad s_2=2.$$

9. Here
$$n=\frac{96}{100}.$$
$$\therefore\ s=\frac{V_1s_1+V_1s_2}{n\,(V_1+V_1)}=\frac{s_1+s_2}{2n}=\frac{1+{\cdot}8}{2n}={\cdot}9\times\frac{100}{96}=\frac{90}{96}=\frac{15}{16}={\cdot}9375.$$

10. Here $$1{\cdot}615=\frac{7\times1{\cdot}843+3\times1}{n(7+3)}.$$

$$\therefore\ n=\frac{15{\cdot}901}{16{\cdot}15}.$$

$$\therefore\ \text{amount of contraction}=10\times(1-n)=\frac{10\times{\cdot}249}{16{\cdot}15}$$

$$=\frac{249}{1615}\text{ cub. cms.}$$

11. Let the amount of A be m lbs.

Then $$s=\frac{m+n}{\frac{m}{s_1}+\frac{n}{s_2}};\quad s'=\frac{m+2n}{\frac{m}{s_1}+\frac{2n}{s_2}};\quad s''=\frac{m+3n}{\frac{m}{s_1}+\frac{3n}{s_2}}.$$

$$\therefore\ m\left(\frac{s}{s_1}-1\right)=n\left(1-\frac{s}{s_2}\right);\quad m\left(\frac{s'}{s_1}-1\right)=2n\left(1-\frac{s'}{s_2}\right),$$

and $$m\left[\frac{s''}{s_1}-1\right]=3n\left[1-\frac{s''}{s_2}\right].$$

Hence by division we have the required equations.

EXAMPLES. IV. (Pages 34—36.)

1. $p=wh=\frac{1000}{1728}\times5280\times12$ ozs. wt. per sq. in. $=2291\frac{2}{3}$ lbs. wt.

2. If x be the depth in inches, then

$$100=15+\frac{1000}{1728\times16}\cdot x.$$

$$\therefore\ x=\frac{85\times1728\times16}{1000}\text{ ins.}=195{\cdot}84\text{ ft.}$$

3. 12090 ozs. = press. per sq. ft. $=1{\cdot}56\times1000\times x$, where x is the required depth in feet.

$$\therefore\ x=\frac{12090}{1560}=7\tfrac{3}{4}\text{ ft.}$$

4. x being the ht. in inches,

$$34=18+w\,.\,x=18+\frac{125}{2}\times\frac{1}{1728}\times x.$$

$$\therefore\ x=\frac{16\times2\times1728}{125}\text{ inches}=\frac{16\times2\times144}{125}\text{ ft.}=36{\cdot}864\text{ ft.}$$

5. x being the required height in inches, then

$$14 = x \times \cdot 00125 \times \frac{62\frac{1}{2}}{1728} = x \times \frac{1}{800} \times \frac{125}{2 \times 1728}.$$

$$\therefore\ x = \frac{14 \times 800 \times 3456}{125} \text{ ins.} = \frac{14 \times 800 \times 288}{125} \text{ ft.}$$

$$= 4 \text{ miles } 1561 \cdot 6 \text{ yds.}$$

6. Force $= \frac{16}{12} \times 1 \times w \,.\, 34 = \frac{4}{3} \times \frac{125}{2} \times 34$ lbs. wt.

$$= 2833\tfrac{1}{3} \text{ lbs. wt.}$$

7. If x be the depth in feet, then

$$x + 30 = 4\,(2 + 30).$$
$$\therefore\ x = 98.$$

8. If the depth required be x ft., then

$$x + 34 = 2\,(10 + 34).$$
$$\therefore\ x = 54.$$

9. Let the atmospheric pressure be that which would be due to a depth x of water.

$$\therefore\ x + 5 = \frac{1}{2}(44 + x).$$
$$\therefore\ x = 34.$$

$\therefore$ required pressure $= 34 \times 62\frac{1}{2}$ lbs. wt. per sq. ft.

$$= 34 \times \frac{125}{2} \times \frac{1}{144} \text{ lbs. wt. per sq. in.}$$

$$= \frac{17 \times 125}{144} = 14\tfrac{109}{144} \text{ lbs. wt. per sq. in.}$$

10. Press. per sq. yd. at a depth of 20 yds.

$$= 20 \times 1 \cdot 026 \times 62\tfrac{1}{2} \times 27 \text{ lbs. wt.}$$

$$= 20 \times \frac{1026}{1000} \times \frac{125}{2} \times \frac{27}{2240} \text{ tons' wt.}$$

$$= 15 \cdot 45 \ldots \text{ tons' wt.}$$

11. Ans. $= \frac{500}{13 \cdot 596}$ metres $= 36 \cdot 77\ldots$ metres.

12. If x be the depth in cms., then

$$1000 = 13 \cdot 596 \times x, \quad i.e.\ x = 73 \cdot 55\ldots.$$

13. Press. per sq. cm. = wt. of 75 c.c. of mercury

$$= 75 \times 13{\cdot}6 \text{ grammes' wt.}$$

Area of valve = 100 sq. cms.

$$\therefore \text{ thrust required} = 75 \times 13{\cdot}6 \times 100 = 102000.$$

14. Press. at depth of 10 feet

$$= 15 + 120 \times \frac{62\frac{1}{2}}{1728} = 15 + \frac{625}{144} = 19\tfrac{49}{144} \text{ lbs. wt.} = 19{\cdot}34\ldots \text{ lbs. wt.}$$

$$\text{Press. at depth of one mile} = 15 + 5280 \times 12 \times \frac{62\frac{1}{2}}{1728}$$
$$= 2306{\cdot}\dot{6} \text{ lbs. wt.}$$

15. $$p = [30 \times 13{\cdot}6 + 24] \times \frac{62\frac{1}{2}}{1728} \text{ lbs. wt.}$$
$$= \frac{125}{8} = 15\tfrac{5}{8} \text{ lbs. wt. per sq. in.}$$

16. $$p = [2{\cdot}5 + 6 \times {\cdot}92] \times 252 \text{ grains' wt. per sq. in.}$$
$$= 2021{\cdot}04 \text{ grains' wt.}$$

17. $$p = [2 \times 13{\cdot}568 + 24] \times \frac{1000}{1728} \times \frac{1}{16} = 1\tfrac{367}{432}.$$

18. If the level of the mercury rises one inch in the smaller tube, it must fall $\frac{1}{10}$ inch in the larger one. Hence, if x be the required height of the water poured in, then

$$xw = \text{press. at lowest point of the water}$$
$$= \text{pressure at the same height in the other tube}$$
$$= \frac{11}{10} \times w \times 13{\cdot}596.$$

Hence $x = 14{\cdot}9556$ ins.

Hence amount poured in $= 14{\cdot}9556$ cub. ins.

20. With a figure similar to that of Ex. 18, we have

$$\rho . DM = \rho . DN + \sigma . NL.$$
$$\therefore \ \rho . MN = \sigma . NL,$$

i.e. $$\rho[\cos\theta - \cos(90^\circ - \theta)] = \sigma[\cos\theta + \cos(180^\circ - \alpha - \theta)].$$
$$\therefore \ \rho(\cos\theta - \sin\theta) = \sigma[\cos\theta - \cos\overline{\alpha + \theta}].$$
$$\therefore \ \rho[1 - \tan\theta] = \sigma[1 - \cos\alpha + \sin\alpha\tan\theta].$$
$$\therefore \ \tan\theta = \text{etc.}$$

21. Let AOB be the horizontal diameter of the tube and C its lowest point. Let D be the point of juncture of the two liquids. Draw DM perpendicular to OC and let $\angle COD=\theta$. Then

$$2\rho\,.\,a=\text{press. at } C=3\rho\,.\,CM+\rho\,.\,OM$$
$$=3\rho\,(a-a\cos\theta)+\rho\,.\,a\cos\theta.$$
$$\therefore\;\; 2=3-3\cos\theta+\cos\theta.$$
$$\therefore\;\; \cos\theta=\frac{1}{2}\text{ and }\theta=60^\circ.$$

Hence $\qquad DOA=30^\circ. \quad \therefore$ arc $CD=$twice arc DA.

22. Let AOB be the bounding diameter, O being the centre. Let the arcs AC, CD, DF, FB be those with liquids of density ρ, 4ρ, 8ρ and 7ρ. Let E be the lowest point, and let $\alpha=45^\circ$. Then, if AOB be inclined at θ to the vertical,

$$\angle AOC=\angle COD=\alpha,\quad \angle DOE=180^\circ-2\alpha-\theta,$$
$$\angle EOF=\theta-\alpha,\quad \angle FOB=\alpha.$$

$$\therefore\;\; 7\rho\,[a\cos(\theta-\alpha)-a\cos\theta]+8\rho\,(a-a\cos\overline{\theta-\alpha})=\text{press. at } E$$
$$=8\rho\,[a-a\cos\overline{180^\circ-2\alpha-\theta}]$$
$$+4\rho\,[\cos\overline{180^\circ-2\alpha-\theta}-\cos\overline{180^\circ-\alpha-\theta}]$$
$$+\rho\,[\cos\overline{180^\circ-\alpha-\theta}-\cos(180^\circ-\theta)].$$
$$\therefore\;\; 8-7\cos\theta-\cos(\theta-45^\circ)=8-4\sin\theta+3\cos(45^\circ+\theta)+\cos\theta.$$
$$\therefore\;\; 0=8\cos\theta-4\sin\theta+\cos(\theta-45^\circ)+3\cos(\theta+45^\circ)$$
$$=8\cos\theta-4\sin\theta+4\cos\theta\cos 45^\circ-2\sin\theta\sin 45^\circ.$$
$$\therefore\;\; \tan\theta=\frac{8+4\cos 45^\circ}{4+2\sin 45^\circ}=2.$$

23.
$$p=[h\rho+h\,.\,2\rho+h\,.\,3\rho+\;\dots\;+h\,.\,n\rho]\,g$$
$$=gh\,.\,\rho\,(1+2+3+\;\dots\;+n)=\frac{n\,(n+1)}{2}\,h\rho g.$$

24. Consider a vertical cylinder of depth h and small cross-section α. Divide it into a very large number n of horizontal sections; let p_{r-1} be the pressure at the bottom of the $(r-1)^{\text{th}}$ of these, and p_r at the bottom of the r^{th}. Since the density of this r^{th} section is very approximately constant and equal to $\frac{\rho}{a}\,.\,\frac{r}{n}\,h$, we have

$$p_n - p_{n-1} = g \cdot \frac{\rho}{a} \cdot \frac{n}{n} h \times \frac{h}{n},$$

$$p_{n-1} - p_{n-2} = g \cdot \frac{\rho}{a} \cdot \frac{n-1}{n} h \times \frac{h}{n},$$

....

$$p_r - p_{r-1} = g \cdot \frac{\rho}{a} \cdot \frac{r}{n} h \times \frac{h}{n},$$

....

$$p_1 - \Pi = g \cdot \frac{\rho}{a} \cdot \frac{1}{n} h \times \frac{h}{n},$$

$\therefore$ by addition $p_n - \Pi = g \cdot \frac{\rho}{a} \cdot \frac{h}{n} [1 + 2 + \ldots + n] \times \frac{h}{n}$

$$= g \cdot \frac{\rho}{a} \cdot \frac{h}{n} \frac{n(n+1)}{2} \times \frac{h}{n} = \frac{1}{2} g \frac{\rho h^2}{a} \left[1 + \frac{1}{n}\right] = \frac{1}{2} g\rho \frac{h^2}{a},$$

when n is made indefinitely great.

EXAMPLES. V. (Pages 41—44.)

1. The depth of the centre of gravity of the face $= 3$ ft. Hence the thrust

$$= 2^2 \times 3 \times w = 12 \times 62\tfrac{1}{2} \text{ lbs. wt.} = 750 \text{ lbs. wt.}$$

2. Thrust $= \dfrac{1\frac{1}{2}}{144} \times 250 \times 62\tfrac{1}{2}$ lbs. wt. $= 162\tfrac{73}{96}$ lbs. wt.

3. Depth of C.G. of vertical face $= 90$ cms.

$\therefore$ thrust $= 30^2 \times 90$ grammes' wt. $= 81000$ grammes' wt.

Depths of C.G. of the upper and lower faces are 75 and 105 cms. Hence the thrusts are

$30^2 \times 75$ and $30^2 \times 105$, *i.e.* 67500 and 94500 grammes' wt.

4. Thrust $= \left(\dfrac{1}{2}\right)^2 \times 20 \times 64$ lbs. wt. $= 320$ lbs. wt.

5. Thrust

$$= 8 \times 12 \times [6 + 33] \times 62\tfrac{1}{2} \text{ lbs. wt.} = 104\tfrac{13}{28} \text{ tons' wt.}$$

6. Thrust

$$= 15^2 \times (15 + 7{\cdot}5) \text{ grammes' wt.} = 5062{\cdot}5 \text{ grammes' wt.}$$

7. Thrust

$=\pi \,.\, 7^2 \times (5000+1033)$ grammes' wt. $=295617\pi$ grammes' wt.

8. Thrust

$$=600 \times \frac{30}{\sin 30^\circ} \times 15 \times w = 600 \times 60 \times 15 \times 62\tfrac{1}{2} \text{ lbs. wt.}$$

$$=\frac{600 \times 60 \times 15 \times 125}{2240 \times 2} \text{ tons' wt.} = 15066\tfrac{27}{28} \text{ tons' wt.}$$

This thrust is independent of the extent of surface of the reservoir.

9. Pressure $=1 \times w$ per sq. ft. $=\dfrac{62\frac{1}{2}}{144}$ lbs. wt. per sq. in.

$$=\frac{125}{288} \text{ lbs. wt. per sq. inch;}$$

whole thrust $=\pi \,.\, 4^2 \,.\, \dfrac{125}{288} = \dfrac{125\pi}{18} = 21\tfrac{52}{63}$ lbs. wt.

10. Let a be the side of the square, and let x be the depth of the horizontal line. Then $a \,.\, x \,.\, \frac{x}{2} w =$ half thrust on the whole square

$$=\frac{1}{2} \,.\, a \,.\, a \,.\, \frac{a}{2} \,.\, w.$$

$$\therefore\ x = \frac{a}{\sqrt{2}}.$$

$$\therefore\ \frac{x}{a-x} = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1.$$

11. The thrust is the same as if there were two liquids, one of depth 10 cms. and of sp. gr. 1, and the other of depth $\frac{10}{3}$ cms. and sp. gr. 12·6.

$$\therefore\ \text{thrust} = 10\left[10 \times 5 + \frac{10}{3} \times \frac{10}{6} \times 12{\cdot}6\right] \text{grammes' wt.}$$

$$=1200 \text{ grammes' wt.} = 1{\cdot}2 \text{ kilog. wt.}$$

12. As in Ex. 11 the thrust

$$=\frac{5}{6} \,.\, 1 \times \frac{1}{2} \times w + \frac{5}{6} \times \frac{2}{3} \times \frac{1}{3} \times 12{\cdot}596 \times w + 10 \,.\, 12 \,.\, 15 \text{ lbs. wt.}$$

$$=\frac{296{\cdot}92}{108} \times 62\tfrac{1}{2} + 1800 \text{ lbs. wt.}$$

$$=1971{\cdot}8\ldots \text{ lbs. wt.}$$

13. Thrust $=2\pi\cdot\frac{1}{3}\cdot 4\cdot 2\cdot w+2\pi\cdot\frac{1}{3}\cdot 2\cdot 1\times 12{\cdot}5w$

$$=\frac{4}{3}\cdot\pi\cdot\frac{33}{2}\cdot\frac{125}{2}\text{ lbs. wt.}=1375\pi\text{ lbs. wt.}$$

$$=4321\tfrac{3}{7}\text{ lbs. wt.}$$

14. Let x and y be the depths of the areas in feet.

Then $$x=4y \text{ and } x-1=9(y-1).$$

$$\therefore\ x=\frac{32}{5}\text{ and } y=\frac{8}{5}.$$

15. Upward thrust on the lid $=1^2\times 20w=1250$ lbs. wt. Downward thrust on base $=1^2\times 21w=1312\frac{1}{2}$ lbs. wt. The thrust of the base is greater than the wt. of the water because it has to balance the downward thrust of the lid on the water as well as the weight of the water.

16. Total thrust on the embankment $=300\times 88\times 44w$

$$=\frac{300\times 88\times 44}{27}\times\frac{3}{4}\text{ tons' wt.}=32266\tfrac{2}{3}\text{ tons' wt.}$$

Total weight of the water

$$=440\times 100\times\frac{44}{3}\times\frac{3}{4}\text{ tons}=484000\text{ tons.}$$

17. If x be the depth in feet, then

$$6\times\frac{1}{2}\cdot(\sqrt{3})^2\sin 60^\circ\times x=\sqrt{3}\times x\times\frac{x}{2}.$$

$$\therefore\ x=9\text{ feet.}$$

18. Let ABC be the horizontal face of the tetrahedron, and from the opposite vertex D draw a perpendicular DO upon ABC. Then O is the centre of the base, and

$$AO=\frac{2}{3}\cdot\frac{\sqrt{3}}{2}a=\frac{a}{\sqrt{3}}.$$

$$\therefore\ DO^2=a^2-\frac{a^2}{3}=\frac{2a^2}{3},$$

so that $$DO=\frac{a}{3}\sqrt{6}.$$

The depth below the surface of the liquid of the centre of gravity of a side face then

$$=d+\frac{1}{3}\cdot DO=d+\frac{a\sqrt{6}}{9}.$$

$$\therefore \text{ thrust on each side face} = \frac{1}{2}a^2\frac{\sqrt{3}}{2}\left(d+\frac{a\sqrt{6}}{9}\right)w.$$

Also thrust on horizontal face $= \frac{1}{2}a^2 \cdot \frac{\sqrt{3}}{2} \cdot d \cdot w$.

Also, if AO meet BC in E, then

$$\tan DEO = \frac{DO}{EO} = \frac{a\sqrt{6}}{3} \div \frac{1}{3} \cdot \frac{a\sqrt{3}}{2} = 2\sqrt{2}.$$

Hence resultant thrust on the tetrahedron

$$= 3 \times \frac{a^2\sqrt{3}}{4}\left(d+\frac{a\sqrt{6}}{9}\right) w \times \cos DEO - \frac{a^2\sqrt{3}}{4}dw$$

$$= 3 \times \frac{a^2\sqrt{3}}{4}\left(d+\frac{a\sqrt{6}}{9}\right) w \times \frac{1}{3} - \frac{a^2\sqrt{3}}{4}dw$$

$$= \frac{a^2\sqrt{3}}{4} \times \frac{a\sqrt{6}}{9} \times w = \frac{a^3 \cdot \sqrt{2}w}{12}.$$

19. If x be the depth in feet, then

$$50 \times 25 \times \frac{25}{2} \times 1{\cdot}026 = 50 \times x \times \frac{x}{2}.$$

$$\therefore\ x^2 = 25^2 \times 1{\cdot}026.$$

$$\therefore\ x = 25 \times \sqrt{1{\cdot}026} = 25 \times 1{\cdot}0129\ldots = 25{\cdot}322\ldots \text{ ft.}$$

20. Let x be the length of the axis occupied by the upper liquid, and h the total length of the axis, so that

$$x^3 = \frac{1}{2}h^3, \text{ and } \therefore\ x = \frac{1}{2} \cdot \sqrt[3]{4} \times h.$$

$$\therefore\ \text{thrust on base} = \pi r^2 \times [3(h-x) + 1 \cdot x]\rho.$$

$$\therefore\ \frac{\text{actual thrust}}{\text{thrust when cone is filled with lighter liquid}}$$

$$= \frac{3h-2x}{h} = 3 - \sqrt[3]{4}.$$

21. Depth of C.G. $= c + \frac{b}{2}\cos\theta$.

$$\therefore\ \text{thrust} = ab\left(c + \frac{b}{2}\cos\theta\right) \cdot w.$$

22. Depth of the C.G. of the upper end

$$= c - \frac{h}{2}\cos\theta,$$

and that of the lower

$$=c+\frac{h}{2}\cos\theta.$$

Hence the thrusts are

$$\pi a^2\left(c-\frac{h}{2}\cos\theta\right)w \text{ and } \pi a^2\left(c+\frac{h}{2}\cos\theta\right)w.$$

23. If a be the radius of the base, and h be the height of the cone, the thrust on the base

$$=\pi a^2\times a\sin(90^\circ-\alpha)\,.\,w=\pi a^3\cos\alpha w.$$

$$\therefore \frac{\text{thrust}}{\text{wt. of contained liquid}}=\frac{\pi a^3\cos\alpha w}{\frac{1}{3}\pi a^2\,.\,hw}=3\cos\alpha\,.\,\tan\alpha=3\sin\alpha.$$

24. The centre of gravity G of the liquid is vertically below the point A of attachment. Hence if V be the vertex of the cone, and C the middle point of the base, we have, if θ be the inclination of the base to the vertical,

$$\tan\theta=\frac{CG}{AC}=\frac{\frac{1}{4}h}{a}=\frac{1}{4}\cot\alpha.$$

$$\therefore \frac{\text{thrust}}{\text{wt. of the water}}=\frac{\pi a^2\,.\,a\cos\theta\,.\,w}{\frac{1}{3}\pi a^2\,.\,hw}=3\tan\alpha\cos\theta$$

$$=\frac{3\tan\alpha}{\sqrt{1+\tan^2\theta}}=\frac{12\tan\alpha}{\sqrt{16+\cot^2\alpha}}=\frac{12\sin^2\alpha}{\cos\alpha\sqrt{16\sin^2\alpha+\cos^2\alpha}}$$

$$=\frac{12\sin^2\alpha}{\cos\alpha\sqrt{15\sin^2\alpha+1}}.$$

25. Let A be the point of attachment, ACB the diameter of the base through A, G the centre of gravity of the liquid, and $\angle CAG=\theta$. Then

$$\tan\theta=\frac{CG}{CA}=\frac{3}{8}.$$

$$\therefore \frac{\text{thrust}}{\text{wt. contained liquid}}=\frac{\pi a^2\,.\,a\cos\theta\,.\,w}{\frac{2}{3}\pi a^3 w}=\frac{3}{2}\cos\theta$$

$$=\frac{3}{2}\frac{1}{\sqrt{1+\left(\frac{3}{8}\right)^2}}=\frac{12}{\sqrt{73}}.$$

26. Let x_r be the depth of the r^{th} dividing line, the depth of the lowest side being a. Then thrust on surface to depth x_r

$$=\frac{r}{n}\times\text{thrust on whole parallelogram.}$$

$$\therefore\ x_r\times\frac{x_r}{2} : a\times\frac{a}{2} :: r : n.$$

$$\therefore\ x_r=a\sqrt{\frac{r}{n}},\quad i.e.\ x_r\propto\sqrt{r}.$$

27. Let the dividing line AX cut DC in X.
Then thrust on ADX=half that on $ADCB$.

$$\therefore\ \triangle ADX\times\frac{2}{3}AD\times w=\frac{1}{2}\times AD\,.\,DC\times\frac{1}{2}AD\times w.$$

$$\therefore\ \frac{1}{2}AD\,.\,DX\,.\,\frac{2}{3}AD\,.\,w=\frac{1}{2}\,.\,AD\,.\,DC\,.\,\frac{1}{2}AD\,.\,w.$$

$$\therefore\ DX=\frac{3}{4}DC.$$

28. Let $ABCD$ be the square, the side AB being in the surface. Let the required line cut AD in P and DC in Q, and let $AD=x$. Then

$$\text{area }\triangle PDQ\,.\,\text{depth of its C.G.}\times w=\frac{1}{2}\,.\,\text{thrust on }ABCD.$$

$$\therefore\ \frac{1}{2}(a-x)^2\,.\left[x+\frac{2}{3}(a-x)\right].\,w=\frac{1}{2}a^2\,.\,\frac{a}{2}\,.\,w.$$

$$\therefore\ 2x^3-6a^2x+a^3=0,$$

an equation to give x.

29. Let AOB be the diameter in the surface; OP_r the bounding line of the r^{th} sector. Then

$$\text{thrust on }AOP_r=\frac{r}{n}\times\text{thrust on the semi-circle}$$

$$=\frac{r}{n}\times\frac{1}{2}\pi a^2\times\frac{4a}{3\pi}w.\qquad\text{[\textit{Statics}, Art. 118.]}$$

Hence, by the same article,

$$\frac{1}{2}a^2\,.\,\angle AOP_r\times\frac{2}{3}\frac{a\sin\frac{1}{2}AOP_r}{\frac{1}{2}\,.\,AOP_r}\sin\frac{AOP_r}{2}w=\frac{r}{n}\times\frac{\pi a^2}{2}\times\frac{4aw}{3\pi}.$$

$$\therefore\ \sin^2\frac{AOP_r}{2}=\frac{r}{n},$$

i.e. $$1-\cos AOP_r=\frac{2r}{n}.$$

Draw P_rM_r perpendicular to BOA. Then

$$a-OM_r=\frac{2r}{n}a.$$

$$\therefore\ AM_r=\frac{2r}{n}.a.$$

Hence the construction, since

$$AM_1=\frac{2a}{n},\quad AM_2=\frac{4a}{n},\quad AM_3=\frac{6a}{n}\ldots.$$

30. Let the triangle be ABC, the vertex C being in the surface. Through A draw the dividing line, meeting CB in P, and let x be the depth of P.

Then thrust on $APC=\frac{1}{2}$ thrust on ABC, *i.e.*

$$\frac{1}{2}CA.CP\sin C\times\frac{\alpha+x}{3}\times w=\frac{1}{2}\times\frac{1}{2}CA.CB\sin C\times\frac{\alpha+\beta}{3}\times w.$$

$$\therefore\ 2(\alpha+x)\,CP=(\alpha+\beta)\,CB,$$

i.e. $$2(\alpha+x).x=(\alpha+\beta)\beta;\ \text{an equation for } x.$$

31. By Art. 42, Ex. 3, we have

$$\rho.ab.\frac{a}{2}=\frac{1}{2}\left[\frac{1}{2}\sigma b(b-a)^2+\frac{1}{2}\rho ab(2b-a)\right].$$

Hence, etc.

32. Let the vertical line through C cut AB in D. We then have required ratio

$$=\frac{\triangle ACD\times\text{depth of its c.g.}\times w}{\triangle BCD\times\text{depth of its c.g.}\times w}$$

$$=\frac{AC\times\frac{1}{3}(AC\cos ACD+CD)}{BC\times\frac{1}{3}(BC\cos BCD+CD)}\qquad\text{[\textit{Statics}, Art. 104.]}$$

$$=\frac{b}{a}.\frac{b\cos\frac{C}{2}+CD}{a\cos\frac{C}{2}+CD}.$$

Now from the triangles ACD, BCD we have

$$\frac{CD}{\sin B}=\frac{DB}{\sin\frac{C}{2}} \text{ and } \frac{CD}{\sin A}=\frac{AD}{\sin\frac{C}{2}},$$

and also
$$\frac{AD}{b}=\frac{DB}{a}=\frac{c}{a+b}.$$

$$\therefore \text{ required ratio}=\frac{b}{a}\frac{b\sin C+2\sin B\,.\,DB}{a\sin C+2\sin A\,.\,DA}$$

$$=\frac{b^2}{a^2}\cdot\frac{c+2DB}{c+2DA}=\frac{b^2}{a^2}\cdot\frac{c\,(a+b)+2ac}{c\,(a+b)+2bc}=\frac{b^2}{a^2}\cdot\frac{3a+b}{a+3b}.$$

33. Let a be the thickness of each liquid so that na is the side. Hence thrust on the base

$$=n^2a^2g\,[n\rho a+\overline{n-1}\,\rho a+\ldots\ldots+2\rho a+\rho a]$$

$$=n^2a^2g\,.\,\rho a\,.\,\frac{n\,(n+1)}{2}\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Now the whole of the liquid above the lower thickness may be replaced by a thickness x of density $n\rho$ where

$$x\times n\rho=a\,(n-1)\,\rho+a\,(n-2)\,\rho+\ldots\ldots+a\rho$$

$$=a\rho\,\frac{n\,(n-1)}{2},$$

so that
$$x=\frac{n-1}{2}\,a.$$

Hence the total thrust on the portion of a side in contact with the lowest liquid

$$=na\times a\times\left[x+\frac{a}{2}\right]\times n\rho g$$

$$=na^2\times\frac{n}{2}\,a\times n\rho g\,\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Clearly (1) is $(n+1)$ times (2).

34. The whole pressure is that due to a liquid of density ρ' in contact with the whole curved surface and to one of density $\rho-\rho'$ in contact with the lower half.

Hence required ratio

$$=\frac{\pi r^2\left[\frac{h\rho}{2}+\frac{h\rho'}{2}\right]g}{2\pi rh\,.\,\frac{h}{2}\rho' g+2\pi r\,\frac{h}{2}\cdot\frac{h}{4}\,(\rho-\rho')\,g}=\frac{2r\,(\rho+\rho')}{h\,(\rho+3\rho')},$$

on reduction.

35. If h be the height of the cone and y the depth of the required cutting plane, then whole press. on the upper half = half that on the lower half.

Hence, since the areas of similar cones are as the squares of their heights,

$$y^2 \times \frac{2y}{3} w = \frac{1}{2} h^2 \times \frac{2h}{3} w.$$

$$\therefore \ y^3 = \frac{4h^3}{8}, \qquad i.e. \ y = \frac{h}{2} \sqrt[3]{4}.$$

In the second case the whole pressure on the lower half is equal to the same.

$$\therefore \ (h-y)^2 \times \left[y + \frac{h-y}{3}\right] = \frac{1}{2} h^2 \times \frac{h}{3},$$

i.e.
$$4y^3 - 6hy^2 + h^3 = 0,$$

i.e.
$$(2y - h)(2y^2 - 2hy - h^2) = 0.$$

$$\therefore \ y = \frac{h}{2}, \qquad \frac{1+\sqrt{3}}{2} h, \qquad \text{or} \qquad \frac{1-\sqrt{3}}{2} h.$$

The first value only is admissible, the second being $> h$, and the third negative.

36. Let the thickness of each section of liquid be h. Consider the r^{th} section. All the sections above it may be replaced by a thickness x of density $r\rho$, where

$$x \cdot r\rho = h[\overline{r-1}\rho + \overline{r-2}\rho + \ldots + \rho] = h\frac{r(r-1)}{2}\rho,$$

so that
$$x = (r-1)\frac{h}{2}.$$

The whole pressure on this r^{th} section then varies as

$$\left(x + \frac{h}{2}\right) \cdot r\rho, \quad i.e. \propto r\rho \times \frac{rh}{2}, \quad i.e. \propto r^2.$$

Hence etc.

EXAMPLES. VI. (Page 47.)

1. Let W be the weight of the lid and a the length of the edge of the box, so that

$$W = \frac{2}{3} a^3 . w \text{ ....(1).}$$

The lid will be on the point of opening if the moment of the weight of the lid about AB is equal to the moment of the thrust on it, which is $wa^2 . \frac{a}{2} \sin 45°$ at a distance $\frac{2}{3} a$ from AB,

i.e. if
$$W \times \frac{a}{2} \cos 45° = \frac{wa^3}{2} \sin 45° \times \frac{2}{3} a,$$
which is true by (1).

2. If a be the side of the box, we have

$$wa^3 = 24.$$

Let x be the required depth of the water, then the thrust on the side is $axw \times \frac{x}{2}$ acting at a distance

$$a - x + \frac{2x}{3}.$$

$$\therefore \frac{x}{2} \times axw \left(a - \frac{x}{3}\right) = 5 \times \frac{a}{2}.$$

$$\therefore x^2 \left(a - \frac{x}{3}\right) \times 24 = 5a^3.$$

$$\therefore 8x^3 - 24ax^2 + 5a^3 = 0.$$

$$\therefore (2x - a)(4x^2 - 10ax - 5a^2) = 0.$$

$\therefore x = \frac{a}{2}$, the other values being found to be inadmissible.

3. The thrust on the lid is $1 . 1 . \frac{1}{2} \sin 45° w$, and acts at a distance $\frac{2}{3}$ ft. from the upper end.

Hence, if W be its weight, on taking moments about the lower edge, we have

$$W \times \frac{1}{2} \cos 45° = 1 . 1 . \frac{1}{2} \sin 45° \times w \times \frac{1}{3}.$$

$$\therefore W = \frac{1}{3} w = \frac{1}{3} \times 62\tfrac{1}{2} \text{ lbs.} = 20\tfrac{5}{6} \text{ lbs.}$$

EXAMPLES. VII. (Pages 53—55.)

1. The depth in each case being the same, the thrusts are as the areas, *i.e.* as $8^2 : 6^2$, *i.e.* as $16 : 9$.

In the first case the pressure of the curved surface on the water has everywhere a downward component. The resultant of all these downward components has to be balanced by the thrust in addition to the weight of the water.

In the second case the same pressure has everywhere an upward component, which helps to support the weight of the water.

2. Let a section through the axis of the wine-glass cut it in VA, VB. Through A, B draw AA', BB' to meet the horizontal plane through V in A', B'. Then total thrust on the base

$$= \text{wt. of water } AA'B'B$$
$$= \pi r^2 \, . \, hw.$$

The thrust on the glass

$$= \text{wt. of water } AA'VB'B$$
$$= \text{wt. of cylinder } AA'B'B - \text{wt. of cone } VAB$$
$$= \pi r^2 hw - \frac{1}{3}\pi r^2 hw = \frac{2}{3}\pi r^2 hw. \quad \therefore \text{ etc.}$$

3. If h' be the height the water occupies in the cylinder, then

$$\frac{1}{3}\pi r^2 h = \pi r^2 \, . \, h', \quad i.e. \quad h' = \frac{1}{3} h.$$

$$\therefore \text{ ratio of the thrusts} = \frac{\pi r^2 h}{\pi r^2 h'} = 3 : 1.$$

4. Let AA', BB' be horizontal and vertical diameters of a section of the cylinder $\perp^{r}$ to its axis. Draw AC, $A'C'$ vertical to meet the horizontal plane through B in C, C'.

Then thrust required

$$= h \times \text{area } CAB'C'C \times w = h \times [ABA' + ACC'A']\, w$$
$$= h \times \left(\frac{1}{2}\pi r^2 + 2r^2\right) w = r^2 hw \, . \left(\frac{\pi}{2} + 2\right).$$

5. Through each point of the circular edge of the base of the hemisphere draw a vertical line to meet the horizontal plane through the highest point of the hemisphere. Then the thrust equals the weight of the water that could be included between the surface thus formed and the hemisphere, and hence

$$= w\left[\pi r^2 \times r - \frac{2}{3}\pi r^3\right] = \frac{1}{3}\pi r^3 w.$$

Also thrust on the base $= \pi r^2 \times r \, . \, w.$

6. Let the horizontal plane through the vertex V and the axis of the cone cut the base in the straight line ACB, C being the centre of the base.

Draw AA', BB', VV' to meet the horizontal plane through the highest point of the cone in A', B', and V'. The thrusts required

$$=w\,[\text{vol. } VBAA'B'V' \mp \text{half volume of cone}]$$

$$=w\,[VV' \times \text{area } \triangle\, V'A'B' \mp \text{half vol. of cone}]$$

$$=w\left[r \times hr \mp \frac{1}{6}\pi r^2 h\right] = whr^2\left[1 \mp \frac{\pi}{6}\right].$$

7. Thrust = wt. of frustum of water − wt. of cylinder, radius $\frac{1}{3}$ ft. and height 1 ft.

$$=\left[\frac{\pi}{3}\left(\frac{1}{3^2}+\frac{1}{3}\cdot\frac{1}{2}+\frac{1}{2^2}\right)-\pi\,.\,\frac{1}{3^2}\right]w \qquad \text{[Page X.}$$

$$=\left[\frac{19\pi}{108}-\frac{\pi}{9}\right]\times\frac{1000}{16}\text{ lbs. wt.} = \frac{875}{216}\,\pi\text{ lbs. wt.}$$

8. Let h and $3h$ be the lengths of the axes of the cylinder and cone. When the cone is uppermost the thrust

$$=w\left[\pi r^2\,.\,3h - \frac{1}{3}\,\pi r^2\,.\,3h\right] = w\pi\,.\,2r^2h.$$

When the cone is lowest the thrust

$$=w\left[\pi r^2\,.\,h + \frac{1}{3}\,\pi r^2\,.\,3h\right] = w\pi\,.\,2r^2h.$$

9. Vertical thrust on the lower cone

$$=\pi r^2\,.\,2hw - \frac{1}{3}\,\pi r^2hw = \frac{5}{3}\,\pi\rho^2h\,.\,w$$

$$=\frac{5}{2}\times\frac{2}{3}\,\pi r^2hw = \frac{5}{2}\times\text{total wt. of the water.}$$

10. Let h and r be the height and radius of the base of the lower cone, h_1 and r_1 those of the upper, so that

$$\frac{h}{r}=\frac{h_1}{r_1}.$$

The upward thrust on the lower cone

$$=\pi r^2\,(h+h_1)\,w - \frac{1}{3}\,\pi r^2hw = \frac{\pi r^2 w}{3}\,(3h_1+2h).$$

The downward thrust on the upper portion

$$=\frac{1}{3}\,\pi r_1^2\,.\,h_1w = \frac{\pi}{3}\,.\,\frac{r^2h_1^3}{h^2}\,w.$$

Hence required upward thrust

$$=\frac{\pi r^2 w}{3}\left[3h_1+2h-\frac{h_1^3}{h^2}\right],$$

and is therefore zero when

$$2h^3+3h_1h^2-h_1^3=0,$$

i.e. when $2h=h_1$.

11. Let V be the vertex and ACB the base of the cone, and let the cup be on the point of rising when the level of the water is $A'C'B'$, where $VC'=h'$ and $VC=h$.

Then

$$\text{wt. of cup}=\pi r^2(h-h')\times w-\text{wt. of } AA'B'B$$

$$=\pi r^2(h-h')\,.\,w-\frac{1}{3}\pi r^2 h\left[1-\frac{h'^3}{h^3}\right].$$

$$\therefore\ \frac{5}{8}\times\frac{1}{3}\pi r^2 hw=\pi r^2(h-h')w-\frac{\pi r^2}{3}\left[h-\frac{h'^3}{h^2}\right],$$

i.e.
$$\frac{5}{24}h=h-h'-\frac{h}{3}+\frac{h'^3}{3h^2}.$$

$$\therefore\ 8h'^3-24h^2h'+11h^3=0.$$

$$\therefore\ (2h'-h)(4h'^2+2hh'-11h^2)=0.$$

$$\therefore\ h'=\frac{h}{2},$$

the other values being inadmissible.

12. Let the cone be on the point of rising when the depth of the surface of the water below the vertex is h'. Then

$$\frac{1}{3}\pi r^2 h\left[1-\frac{h'^3}{h^3}\right]w=\pi r^2(h-h')w-\frac{1}{3}\pi r^2 hw\left[1-\frac{h'^3}{h^3}\right],$$

since the volumes of similar cones are as the cubes of their heights.

$$\therefore\ \frac{2}{3}h\left[1-\frac{h'^3}{h^3}\right]=h-h'\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$\therefore\ 2h'^3-3h^2h'+h^3=0.$$

$$\therefore\ (h'-h)(2h'^2+2hh'-h^2)=0.$$

$$\therefore\ h'=\frac{\sqrt{3}-1}{2}h,$$

the other values being inadmissible.

Hence required ratio

$$=\frac{1}{3}\pi r^2hw\left(1-\frac{h'^3}{h^3}\right):\frac{1}{3}\pi r^2hw=1-\frac{h'^3}{h^3}:1$$

$$=\frac{3}{2}\frac{h-h'}{h}:1, \text{ by equation (1)},$$

$$=\frac{3}{2}\left[1-\frac{\sqrt{3}-1}{2}\right]:1=\frac{9-3\sqrt{3}}{4}:1.$$

13. Wt. of cone $=$ upward thrust on curved surface

$$=\pi r^2.\frac{3h}{2}w-\frac{1}{3}\pi r^2hw-\frac{1}{3}.\frac{1}{8}.\pi r^2hw$$

$$=\pi r^2hw\left[\frac{3}{2}-\frac{1}{3}-\frac{1}{24}\right]=\frac{9}{8}.\pi r^2hw.$$

$$\therefore\ \frac{\text{wt. of either cone}}{\text{wt. of liquid it can hold}}=\frac{\frac{9}{16}\pi r^2hw}{\frac{1}{3}\pi r^2hw}=\frac{27}{16}.$$

EXAMPLES. VIII. (Pages 58, 59.)

1. Horizontal thrust on curved surface

$=$ horizontal thrust on plane base $=\pi a^2.h.w.$

Vertical thrust on it

$=$ wt. of water displaced by the hemisphere $=\frac{2}{3}\pi a^3w.$

$$\therefore\ \text{resultant thrust}=\pi a^2w\sqrt{h^2+\frac{4a^2}{9}}$$

at an angle $\qquad \tan^{-1}\dfrac{\frac{2}{3}\pi a^3w}{\pi a^2hw},$

i.e. $\qquad \tan^{-1}\dfrac{2a}{3h}$ to the horizon.

2. Horizontal thrust

$=$ thrust on vertical section through the axis

$=$ area of section $\times$ depth of its C.G. $\times\, w = rh'\times h\times w.$

3. Horizontal thrust

$=$ thrust on this vertical section $=rh\times\frac{h}{3}w=\frac{1}{3}rh^2w.$

4. Horizontal thrust

$=$ thrust on the vertical section $= h \,.\, 2r \times r \times w = 2r^2hw.$

Vertical thrust $= \frac{1}{2}\pi r^2 \,.\, hw.$

$$\therefore \text{ resultant thrust} = r^2hw\sqrt{4+\frac{\pi^2}{4}}$$

at an angle $$\tan^{-1}\frac{2r^2hw}{\frac{1}{2}\pi r^2hw},$$

i.e. $$\tan^{-1}\frac{4}{\pi}, \text{ with the vertical.}$$

Also, since the pressure at each point of the cylinder is normal and therefore towards the centre of the circular section through the point, each such pressure meets the axis of the cylinder. Hence the resultant thrust must meet the axis, and hence by symmetry must pass through the centre.

5. If h be the height of the cylinder, the vertical thrust

$$= \frac{1}{2}\pi r^2\frac{h}{2}[w+2w] = \frac{3\pi}{4}r^2hw.$$

The horizontal thrust $=$ thrust on the vertical rectangular section $=$ sum of thrusts due to a liquid of sp. gr. unity extending over all the section and to a second liquid of the same sp. gr. extending over the lower half only

$$= 2rh \,.\, \frac{h}{2}w + 2r \,.\, \frac{h}{2} \cdot \frac{h}{4}w = \frac{5}{4}rh^2w.$$

Hence $$\tan\theta = \frac{3\pi}{4}r^2hw \div \frac{5}{4}rh^2w = \frac{3\pi}{5} \cdot \frac{r}{h}.$$

6. If l be the length and a the radius of the cross-section of the trough, then

$$W = \frac{1}{2}\pi a^2lw.$$

Horizontal thrust on either curved surface $=$ thrust on the rectangle of length l and depth a

$$= la \times \frac{a}{2} \,.\, w = \frac{W}{\pi}.$$

Vertical thrust on either half

$$= \frac{1}{4}\pi a^2lw = \frac{W}{2}.$$

$$\therefore\ \tan\theta = \text{horizontal thrust} \div \text{vertical thrust} = \frac{2}{\pi}.$$

$$\therefore\ \theta = \cot^{-1}\frac{\pi}{2}.$$

7. Vertical thrust = wt. of the water displaced by the half-cone

$$= \frac{1}{6}\pi r^2 hw = \frac{1}{6} rhw \,.\, \pi r.$$

Horizontal thrust = thrust on the vertical triangular face

$$= rh \,.\, \frac{h}{3} \,.\, w = \frac{1}{6} rhw \,.\, 2h.$$

$$\therefore\ \text{resultant thrust} = \frac{1}{6} rhw\sqrt{\pi^2 r^2 + 4h^2}.$$

$$\therefore\ \tan\theta = \text{vertical thrust} \div \text{horizontal thrust} = \frac{1}{2}\pi\frac{r}{h} = \frac{\pi}{2}\tan\alpha.$$

8. If σ be the specific gravity of the cone then $w \times$ volume displaced = upward thrust of the water = weight of the cone = $\sigma w \times$ volume of the cone,

i.e. $$h'^3 = \sigma \,.\, h^3 \quad\quad (1),$$

since the volumes of similar cones are as the cubes of their heights.

The centre of gravity of the base of the cone is at a distance $\frac{4a}{3\pi}$ [*Statics*, Art. 118, Cor.] from the centre.

$\therefore$ distance from the axis of the centre of gravity of the half cone

$$= \frac{3}{4} \times \frac{4a}{3\pi} = \frac{a}{\pi}.$$

So distance from the axis of the C.G. of the fluid displaced

$$= \frac{a'}{\pi} = \frac{h'}{h} \,.\, \frac{a}{\pi}.$$

Also horizontal thrust on surface of the cone = thrust on triangular section $= h' \,.\, a' \,.\, \frac{h'}{3} w$, and acts at a distance $\frac{h'}{2}$ from the vertex of the cone.

Hence, since weight of half-cone, or of liquid displaced,

$$= \frac{1}{6}\pi\sigma a^2 hw,$$

we have

$$\frac{h'^2 a'}{3} w \times \frac{h'}{2} + \frac{1}{6}\pi\sigma a^2 hw \times \frac{h'}{h}\frac{a}{\pi} > \frac{1}{6}\pi\sigma a^2 hw \times \frac{a}{\pi},$$

i.e. $$\frac{1}{6} h'^3 a' + \frac{1}{6} a^3 h' \times \frac{h'^3}{h^3} > \frac{1}{6} a^3 h \times \frac{h'^3}{h^3},$$

by equation (1).

Also
$$\frac{a'}{h'}=\frac{a}{h}.$$

$$\therefore\ \frac{h'^4}{h}a+a^3\frac{h'^4}{h^3}>a^3\frac{h'^3}{h^2}.$$

$$\therefore\ h'(a^2+h^2)>a^2h.$$

$$\therefore\ \frac{h'}{h}>\frac{a^2}{a^2+h^2},\ \textit{i.e.}>\frac{\tan^2\alpha}{1+\tan^2\alpha},$$

$$\textit{i.e.}\ >\sin^2\alpha.$$

9. The water will not flow out if the moment of the vertical thrust about the hinge is greater than that of the horizontal thrust.

The vertical thrust $=\frac{1}{6}\pi a^2hw$, and acts at a horizontal distance from the vertex which

$$=\frac{3}{4}\cdot\frac{4a}{3\pi}=\frac{a}{\pi}.$$

The horizontal thrust = thrust on the vertical section $=ahw\cdot\frac{2h}{3}$, and acts at a point $\frac{3h}{4}$ below the vertex.

Hence the water will not flow out if

$$\frac{1}{6}\pi a^2hw\times\frac{a}{\pi}>\frac{2ah^2w}{3}\times\frac{3h}{4},$$

i.e. if $a^2>3h^2$,

i.e. if $\tan^2\alpha>3$,

i.e. if $\tan\alpha>\sqrt{3}$,

i.e. if $\alpha>60°$, and hence vertical angle $>120°$.

EXAMPLES. IX. (Pages 63, 64.)

1. Let the plane base be inclined at α to the horizon so that $\tan\alpha=2$, and therefore $\sin\alpha=\frac{2}{\sqrt{5}}$ and $\cos\alpha=\frac{1}{\sqrt{5}}$.

The resultant thrust on the whole hemisphere is $\frac{2}{3}\pi a^3w$, where a is the radius, and is vertical.

The thrust X on the plane face is $\pi a^2\,.\,a\sin\alpha\,.\,w$. Hence the resultant vertical thrust on the curved surface

$$=\frac{2}{3}\pi a^3w+X\cos\alpha=\frac{2}{3}\pi a^3w+\pi a^3w\,.\,\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}=\frac{16}{15}\pi a^3w.$$

The resultant horizontal thrust

$$=X \sin \alpha = \pi a^3 w \sin^2 \alpha = \frac{4}{5}\pi a^3 w.$$

Hence the required resultant thrust

$$=\frac{4}{5}\pi a^3 w \sqrt{1+\frac{16}{9}} = \frac{4}{3}\pi a^3 w = \text{twice the wt. etc.}$$

2. Let A be the point of attachment, G the centre of gravity. Then in equilibrium AG is vertical.

Hence inclination of the base AB to the vertical

$$=\tan^{-1}\frac{\text{Height of cylinder}}{\text{Diameter of its base}} = 45^\circ.$$

Hence, if $2a$ be the diameter of base or height of cylinder, the thrust X on the upper plane end

$$=\pi a^2 . a \cos 45^\circ w = \frac{\pi a^3 w}{\sqrt{2}}.$$

And thrust Y on the lower plane end

$$=\pi a^2 [a \cos 45^\circ + 2a \sin 45^\circ]\, w = \frac{3\pi a^3 w}{\sqrt{2}}.$$

Also total vertical thrust on whole cylinder = wt. of the water displaced $= \pi a^2 . 2a . w = 2\pi a^3 w$.

Therefore resultant vertical thrust on the curved surface

$$=2\pi a^3 w + X \cos 45^\circ - Y \cos 45^\circ = 2\pi a^3 w - \frac{2\pi a^3 w}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$$

$$=\pi a^3 w = \text{one half the wt. of the contained water.}$$

Also resultant horizontal thrust

$$=Y \cos 45^\circ - X \cos 45^\circ = (Y - X) \times \frac{1}{\sqrt{2}} = \pi a^3 w = \text{etc.}$$

3. If A be the highest point of the rim, O the centre of the plane face, and G be the centre of gravity, then AG must be vertical and

$$\tan GAO = \frac{OG}{OA} = \frac{3}{8}.$$

Hence, if α be the inclination of the plane face to the horizontal,

$$\cos \alpha = \sin GAO = \frac{3}{\sqrt{73}} \text{ and } \sin \alpha = \frac{8}{\sqrt{73}}.$$

Hence, if X be the thrust on the plane face, the $\angle$ required

$$=\tan^{-1}\frac{\text{resultant vertical thrust on the whole hemisphere}+X\cos\alpha}{X\sin\alpha}$$

$$=\tan^{-1}\frac{\frac{2}{3}\pi a^3 w+\pi a^2 . a\sin\alpha . w . \cos\alpha}{\pi a^2 . a\sin\alpha . w . \sin\alpha}$$

$$=\tan^{-1}\frac{\frac{2}{3}+\frac{24}{73}}{\frac{64}{73}}=\tan^{-1}\frac{146+72}{192}=\tan^{-1}\frac{218}{192}=\tan^{-1}\frac{109}{96}.$$

4. If 2α be the vertical angle of the cone, the inclination of the plane base to the vertical is α, and hence the thrust X on the plane base $=\pi a^2 . a\cos\alpha . w$.

Hence required horizontal thrust on curved surface

$$=X\cos\alpha=\pi a^3 w\cos^2\alpha,$$

and required vertical thrust $-X\sin\alpha$

$$=\text{required downward vertical thrust on the whole cone}$$

$$=\text{wt. of contained water}=\frac{1}{3}\pi a^2 hw=W.$$

Therefore required vertical thrust

$$=\frac{\pi}{3}a^2 hw+\pi a^3 w\sin\alpha\cos\alpha$$

$$=W+3W\frac{a\sin\alpha\cos\alpha}{h}=W+3W\sin\alpha\cos\alpha\tan\alpha$$

$$=W(1+3\sin^2\alpha).$$

Also required horizontal thrust

$$=\pi a^3 w\cos^2\alpha=3W\cos^2\alpha . \frac{a}{h}=3W\sin\alpha\cos\alpha.$$

5. The resultant vertical thrust, V, on the whole cone

$$=\text{the wt. of the water displaced}=\frac{1}{3}\pi r^2 h . w.$$

The thrust on the plane base, X,

$$=\pi r^2 . r\cos\alpha . w.$$

The thrust on the curved surface, together with X, gives V.

Hence vertical thrust on curved surface

$$=V-X\sin\alpha,$$

and horizontal thrust $=X\cos\alpha$.

$$\therefore\ \tan\theta = \frac{X\cos\alpha}{V - X\sin\alpha}$$

$$= \frac{\cos\alpha \times r\cos\alpha}{\frac{1}{3}h - r\cos\alpha\sin\alpha} = \frac{3\cos^2\alpha\,.\,h\tan\alpha}{h - 3\cos\alpha\sin\alpha\,.\,h\tan\alpha}$$

$$= \frac{3\sin\alpha\cos\alpha}{1 - 3\sin^2\alpha} = \frac{3\sin\alpha\cos\alpha}{\cos^2\alpha - 2\sin^2\alpha} = \frac{3\tan\alpha}{1 - 2\tan^2\alpha}.$$

6. The cone floats with its axis in the surface and half its volume immersed. The resultant vertical thrust, V, $=$wt. of the liquid displaced

$$= \frac{1}{6}\pi r^2 hw.$$

The thrust, X, on the semi-circular base

$$= \frac{1}{2}\pi r^2 \times \frac{4r}{3\pi} \times w = \frac{2}{3}r^3 w.$$

The resultant thrust on the curved surface, together with X, gives V.

Hence vertical component of this thrust $= V$, and horizontal $= X$,

$$\therefore\ \tan\theta = \frac{X}{V} = \frac{\frac{2}{3}r^3}{\frac{1}{6}\pi r^2 h} = \frac{4r}{\pi h} = \frac{4\tan\alpha}{\pi}.$$

7. The resultant vertical thrust, V, of the water $= \frac{1}{3}\pi r^2 hW$, and is downward.

The thrust on the base, X, $= \pi r^2 \times r\cos(\alpha+\beta) \times w$ and acts at an angle $(\alpha+\beta)$ to the horizon and upwards.

The required thrust on the surface together with X gives V.

The required horizontal component therefore

$$= X\cos(\alpha+\beta) = \pi r^3 w\cos^2(\alpha+\beta)$$

$$= \frac{1}{3}\pi r^2 hw \times 3\tan\alpha\cos^2(\alpha+\beta) = 3W\tan\alpha\cos^2(\alpha+\beta),$$

where W is the weight of the water in the cone.

Also the downward vertical component

$$= V + X\sin(\alpha+\beta) = W + 3W\tan\alpha\,.\,\sin\overline{\alpha+\beta}\cos\overline{\alpha+\beta}$$

$$= W\left[1 + \frac{3}{2}\tan\alpha\sin(2\alpha+2\beta)\right].$$

8. The resultant vertical thrust on each semi-circular end

$$=\text{wt. of the water displaced by the end}=\frac{2}{3}\pi a^3 w.$$

The resultant horizontal thrust $=$ thrust on the vertical circular section forming the base of the hemisphere $=\pi a^2 \times aw$.

Therefore resultant

$$=\pi a^3 w\sqrt{1+\frac{4}{9}}=\frac{\pi}{3}a^3 w\sqrt{13},$$

and is inclined to the horizontal at an angle

$$=\tan^{-1}\frac{\frac{2}{3}\pi a^3 w}{\pi a^3 w}=\tan^{-1}\frac{2}{3}.$$

Also since the pressure at each element of the hemispherical end goes through the centre of the hemisphere, the whole thrust must pass through it also.

9. Let AOB be the diameter of the sphere through the centre O and the lowest point A. Let P be a point on OA such that $OP=\frac{a}{4}$; then by Ex. 2, page 61, P is the centre of pressure of the vertical circular section through A and thus is the point through which passes X, the horizontal thrust of the water on either hemisphere.

Also $X=\pi a^2 . aw$.

Through O draw the radius perpendicular to the given vertical plane, and on it take $OG=\frac{3a}{8}$. Then through G acts $Y\left(=\frac{2}{3}\pi a^3 w\right)$ the resultant vertical thrust on the hemisphere.

Taking moments about A we have, if T be the required tension,

$$T\times 2a=X . AP+Y . OG$$

$$=\pi a^3 w\times\frac{3a}{4}+\frac{2}{3}\pi a^3 w . \frac{3a}{8}=\pi a^4 w.$$

$$\therefore\ T=\frac{\pi a^3 w}{2}=\frac{3}{8} . \frac{4}{3}\pi a^3 w$$

$$=\frac{3}{8}\times\text{given wt. of water.}$$

10. With the notation of the previous article we have, in addition, the weight W' of the hemisphere acting at G', the centre of gravity of the hemispherical shell, which is a point on OG such that $OG'=\frac{1}{2}a$.

Hence, by taking moments about the highest point B, we see that the contact will not give way if

$$W' \times OG' + Y \times OG > X \times BP,$$

i.e. if $$W' \times \frac{a}{2} > \pi a^3 w \times \frac{5a}{4} - \frac{2}{3}\pi a^3 w \times \frac{3a}{8},$$

i.e. $$> \pi a^4 w,$$

i.e. if $$W' > 2\pi a^3 w,$$

i.e. if $$2W' > 3 \times \frac{4}{3}\pi a^3 w,$$

i.e. if wt. of the shell > three times the weight of the contained water.

EXAMPLES. X. (Pages 67—70.)

1. V being the volume, we have

$$160 = \left(V - \frac{4}{1728}\right) \times w = \left(V - \frac{4}{1728}\right) \times 62\tfrac{1}{2},$$

giving $$V = 2\tfrac{6073}{10800}.$$

2. V, V' being the volumes of the iron and cork, we have

$$V' \times \frac{1}{4} \times w = 1, \text{ and } V \times 7 \,.\, w + V' \times \frac{1}{4} \,.\, w = (V + V') \,.\, 1 \,.\, w.$$

$$\therefore\ V \,.\, 6w = V' \times \frac{3}{4} \,.\, w = 3,$$

$$\therefore\ \text{wt. of iron} = V \,.\, 7 \,.\, w = \frac{7}{6} \times 3 \text{ lbs.} = 3\tfrac{1}{2} \text{ lbs.}$$

3. Let V be the volume of the body; its sp. gr. is unity since it just floats in water.

Then $V \,.\, 1 + 42{\cdot}5 = V \times 1{\cdot}85$, since $w = 1$ in the C.G.S. system.

$$\therefore\ V = \frac{42{\cdot}5}{{\cdot}85} = 50 \text{ cub. cms.}$$

4. Let s = sp. gr. of the gas compared with air.

Then $\frac{3}{2}s \,.\, w + 1$ oz. $= \frac{3}{2} \times 1 \,.\, w$. Also $w = 1{\cdot}2$ oz.

$$\therefore\ \frac{3}{2} \,.\, s \,.\, \frac{6}{5} = \frac{3}{2} \,.\, \frac{6}{5} - 1 = \frac{8}{10}.$$

Hence $$s=\frac{8}{18}=\frac{4}{9}=\cdot\dot{4}.$$

Also $$\frac{\text{wt. of a cub. ft. of the gas}}{\text{wt. of a cub. ft. of air}}=\frac{s\times 1\cdot 2\text{ oz.}}{1000\text{ ozs.}}=\frac{8}{15000}=\cdot 0005\dot{3}.$$

5. Here, if V be the volume in cubic decimetres,

$$V\times\cdot 089+50000=V\times 1\cdot 2.$$

$$\therefore\ V=\frac{50000}{1\cdot 111}.$$

$$\therefore\ \text{volume}=V\text{ cub. decimetres}=\frac{50}{1\cdot 111}\text{ cub. metres}$$

$$=45\frac{5}{1111}\text{ cub. metres.}$$

6. V and s being the volume and sp. gr., we have

$$275=Vsw=V\,.\,s\,.\,1,$$

and $$275=\frac{5}{9}\,.\,V\times 15\cdot 59.$$

$$\therefore\ s=\frac{5}{9}\times 15\cdot 59=8\cdot 66\dot{1},$$

and $$V=\frac{275\times 9}{5\times 15\cdot 59}=31\tfrac{1171}{1559}.$$

7. x ft. being the depth below the surface, we have

$$x\times 1\cdot 025=(x+30)\times\cdot 918.$$

$$\therefore\ x=\frac{30\times\cdot 918}{1\cdot 025-\cdot 918}=\frac{27\cdot 54}{\cdot 107}=257\tfrac{41}{107}.$$

8. If V be the volume in cubic feet below the water-line, and x be the number of feet the ship rises, we have

$$V\,.\,1\,.\,w=1000\times 2240=(V-15000\,.\,x)\times 1\cdot 026\,.\,w.$$

$$\therefore\ 1000\times 2240=[1000\times 2240-15000\,.\,x\,.\,w]\times 1\cdot 026.$$

$$\therefore\ x=\frac{1000\times 2240\times\cdot 026}{15000\times w\times 1\cdot 026}=\frac{26\times 2240\times 16}{15000\times 1026}\text{ ft.}$$

$$=\frac{46592}{769500}\text{ ft.}=\cdot 726\ldots\text{ inch.}$$

9. If W be the weight of the ship in tons, and V the original volume below the water-line, then

$$W \times 2240 = V\frac{41}{40}\,.\,w = \left(V + \frac{a}{12}\right).1.w \text{ ....(1)},$$

and $$(W - x) \times 2240 = \left(V + \frac{a-b}{12}\right).1.w \text{ ....(2)}.$$

(1) gives $$V = \frac{40a}{12} \text{ and } w = \frac{12}{41a}\,.\,W.2240.$$

Substituting in (2), we get $W = \dfrac{41ax}{b}$.

10. Originally length immersed $= \cdot 8$ ft. $= \frac{4}{5}$ ft.

For equilibrium after the snow has fallen,

$$\frac{4}{5} \times 1\cdot025w = \frac{4}{5} \times 1.w + x,$$

where x is the wt. of the snow in ozs.

$$\therefore\ x = \left(\frac{4\cdot1}{5} - \frac{4}{5}\right)w = \frac{1}{50} \times 1000 = 20.$$

11. Let V and V' be the volumes of the two portions of wood.

Then $$V \times 1\cdot35w + V' \times \cdot65 \times w = (V + V') \times 1.w.$$

$$\therefore\ V(1\cdot35 - 1) = V'(1 - \cdot65).$$

$$\therefore\ V = V'.$$

12. Let V, V' be the two volumes and s the sp. gr. of the cork. Then

$$V.s.w = 19; \quad V' \times 10\cdot5 \times w = 63,$$

and $$19 + 63 = (V + V').1.w.$$

$$\therefore\ 82 = \frac{19}{s} + \frac{63}{10\cdot5} = \frac{19}{s} + 6. \qquad \therefore\ s = \frac{1}{4}.$$

13. Let x inches be the required length. Then

$$x \times 7\cdot5 \times w + 2 \times 21w = (x+1) \times 13\cdot5 \times w.$$

$$\therefore\ x = \frac{42 - 13\cdot5}{13\cdot5 - 7\cdot5} = \frac{28\cdot5}{6} = 4\cdot75.$$

14. Let V be the apparent volume of the gold and V' the volume of the cavity, if there be one.

Then and $$\left.\begin{aligned}(V - V') \times 19\cdot25 &= 96\cdot25\\ V \times 1 &= 6\end{aligned}\right\}.$$

$$\therefore\ V - V' = 5 \text{ and } V = 6, \text{ so that } V' = 1 \text{ cub. cm.}$$

15. Let V and V' be the volumes of the man and cork. Then $V \times 1{\cdot}1 \times w = 140$ and $(V+V') \times 1 \,.\, w = V \times 1{\cdot}1 \times w + V' \times {\cdot}24 \times w$.

Hence $$Vw = \frac{1400}{11} \text{ and } V' \times {\cdot}76 = V \times {\cdot}1.$$

$$\therefore\; V' = \frac{10}{76} V = \frac{10}{76} \times \frac{1400}{11 \times 62\frac{1}{2}} = \frac{112}{418} \text{ cub. ft.} = 463\tfrac{1}{209} \text{ cub. ins.}$$

16. Let h be the length and $4r$ the external radius of the pencil, and s the sp. gr. of the lead.

Then

$$\pi r^2 h \times s \,.\, w + \pi\,[16r^2 - r^2]\, h \times {\cdot}78 \times w = \pi \,.\, 16r^2 \,.\, \frac{7h}{8} \,.\, 1 \,.\, w,$$

$$\therefore\; s = 14 - 15 \times {\cdot}78 = 14 - 11{\cdot}7 = 2{\cdot}3.$$

17. If s be the sp. gr. of the wood, the sp. grs. of the two liquids are $\frac{5s}{4}$ and $\frac{3s}{2}$.

When the liquids are mixed in equal quantities by weight, the sp. gr. of the mixture

$$= \frac{1+1}{\dfrac{1}{\frac{5s}{4}} + \dfrac{1}{\frac{3s}{2}}}, \text{ by Art. 25,}$$

$$= \frac{15s}{11}.$$

Hence if x be the required fraction,

$$1 \times s = x \times \frac{15s}{11},$$

so that $$x = \frac{11}{15}.$$

18. If s be the sp. gr. of the solid, those of the liquids are as, bs, cs.

When equal volumes are mixed the sp. gr.

$$= \frac{a+b+c}{3} s, \qquad \text{[Art. 24.}$$

and the required fraction

$$= \frac{3}{a+b+c}.$$

When equal wts. are mixed, the sp. gr.

$$= \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}, \qquad \text{[Art. 25.}$$

and the required fraction

$$=\frac{1}{3}\left[\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right].$$

19. The weight of the second mass in water must be 26 lbs. Hence, if x be this weight,

$$x\left(1-\frac{1}{7\cdot 5}\right)=26.$$

$$\therefore\ x=\frac{7\cdot 5}{6\cdot 5}\times 26=30 \text{ lbs.}$$

20. If W be the weight of the box and x the depth required in feet, then

$$W=\frac{3\frac{3}{4}}{12}\,.\,1^2\,.\,w,$$

and
$$x\,.\,1^2\,.\,w=W+\left(x-\frac{1}{12}\right)\times\left(\frac{5}{6}\right)^2.\,w.$$

$$\therefore\ x=\frac{15}{48}+\left(x-\frac{1}{12}\right)\times\frac{25}{36},$$

so that
$$x=\frac{5}{6}\text{ ft.}$$

Hence amount poured in

$$=\left(x-\frac{1}{12}\right)\times\left(\frac{5}{6}\right)^2\text{ cub. ft.}=\frac{3}{4}\times\frac{25}{36}\text{ cub. ft.}=900\text{ cub. ins.}$$

21. The C.G. of P and W must be at the C.G. of the displaced fluid. Hence

$$\frac{P\times 0+W\times\frac{1}{2}}{P+W}=\frac{1}{2}\left(1-\frac{1}{n}\right)=\frac{n-1}{2n}.$$

$$\therefore\ W=P\,(n-1).$$

22. Here $n=2$ and thus, as in the last example,

$$W=P\,(2-1)=P.$$

23. As in Ex. 21,

$$W=P\,(2-1)=P;$$

also the upward thrust of the displaced water must $=W+P$,

$$\therefore\ W+P=\frac{1}{2}\,.\,\frac{W}{s};\quad\therefore\ s=\frac{1}{4}.$$

24. As in Exs. 21, 23, if $2a$ be the length of the rod and k its section,

$$\frac{P.0+2ak\rho.a}{P+2ak\rho}=\frac{1}{2}.x,$$

where x is the length immersed, and $P=\frac{1}{n}.2ak\rho$,

i.e.
$$x=\frac{2a}{1+\frac{1}{n}}=\frac{n}{n+1}.2a,$$

and
$$\left(1+\frac{1}{n}\right).2ak\rho=x\times\sigma\times k.$$

$$\therefore \frac{n+1}{n}\rho=\frac{x\sigma}{2a}=\frac{n}{n+1}\sigma,$$

i.e.
$$(n+1)^2.\rho=n^2.\sigma.$$

25. As the bottle descends the air is compressed and occupies less and less volume so that the water displaced decreases and therefore the upward thrust diminishes. Finally there will be a position where the upward thrust of the water just equals the wt. of the bottle, and here it will neither sink nor rise. If immersed to a greater depth than this position it will sink.

26. Let A square feet be the area of the water-section of the ship, so that

$$A\times\frac{1}{12}\times w=30\times 2240 \text{.....................}(1).$$

Let W be the weight of the ship and V the original volume displaced, so that

$$W=V.w \text{..................................}(2)$$

and
$$W-600\times 2240=(V-2A)\times w'=(V-2A).64 \text{ ....}(3).$$

Substituting in (3) from (1) and (2), we have

$$W-600\times 2240=\frac{64}{62\frac{1}{2}}W-2\times 64\times\frac{30\times 2240\times 12}{62\frac{1}{2}}.$$

$$\therefore \frac{3W}{125}=\left(\frac{4\times 64\times 30\times 2240\times 12}{125}-600\times 2240\right)\text{lbs.}$$

$$\therefore W=\left(\frac{4\times 64\times 30\times 12}{3}-200\times 125\right)\text{tons}$$

$$=(30720-25000)\text{ tons}=5720\text{ tons.}$$

27. If x be the length out of the liquid, then, since the volumes of similar cones are as the cubes of their heights,

$$\therefore h^3\times\rho=(h^3-x^3)\sigma.$$

$$\therefore x=h\sqrt[3]{1-\frac{\rho}{\sigma}}.$$

28. Volume of the cone $=\frac{1}{3}\pi.1^2.7=\frac{7}{3}\pi$.

Volume of the portion of it out of the liquid $=\left(\frac{3}{7}\right)^3\times\frac{7\pi}{3}=\frac{9\pi}{49}$.

Volume of the hemisphere $=\frac{2}{3}\pi$.

Hence $$\frac{2\pi}{3}\times\frac{7}{4}+\frac{7\pi}{3}\times\frac{3}{2}=\left(\frac{7\pi}{3}-\frac{9\pi}{49}+\frac{2\pi}{3}\right)\sigma.$$

$$\therefore \frac{7}{6}+\frac{7}{2}=\left(3-\frac{9}{49}\right)\sigma=\frac{138}{49}\sigma.$$

$$\therefore \sigma=\frac{343}{207}.$$

29. If the axis of the cone be in the surface of the liquid, then half the volume of the cone is in the liquid and hence the upward thrust of the liquid just balances the weight of the cone. Also the centres of gravity of the cone and liquid displaced are clearly in the same vertical line. $\therefore$ etc.

30. Let 2α be the vertical angle of the cone, x the depth originally immersed, and W the weight of the cone. Then

$$W=\frac{1}{3}\pi x^3\tan^2\alpha . w,$$

and $$W+\frac{1}{3}\pi x^3\tan^2\alpha w=\frac{1}{3}\pi h^3\tan^2\alpha . w.$$

$$\therefore 2x^3=h^3,$$

i.e. $$x^3=\frac{4h^3}{8},\ \textit{i.e. } x=\frac{h}{2}\times\sqrt[3]{4}.$$

EXAMPLES. XI. (Pages 71, 72.)

1. Here $$h.\sigma=\frac{h}{2}.1+\frac{h}{2}\times{\cdot}0013.$$

$$\therefore \sigma=\frac{1{\cdot}0013}{2}={\cdot}50065.$$

2. x being the length in the lower liquid,

$$1\times1{\cdot}2=x\times1{\cdot}5+(1-x)\times1.$$

$$\therefore {\cdot}2=x\times{\cdot}5,\ \textit{i.e. } x=\frac{2}{5}.\quad\therefore \text{etc.}$$

3. Here $$5{\cdot}1432\times\sigma=5{\cdot}0697\sigma+1\times 1.$$

$$\therefore\ \sigma=\frac{1}{\cdot 0735}=\frac{2000}{147}=13\tfrac{89}{147}=13{\cdot}6054\ldots.$$

4. V, V' being the volumes of gold and silver, we have

$$V\times 19{\cdot}25\times w+V'\times 10{\cdot}5\times w$$
$$=\frac{15}{16}(V+V')\times 13{\cdot}6\times w+\frac{1}{16}(V+V')\times 1\times w.$$

$$\therefore\ V\left[19{\cdot}25-\frac{15}{16}\times 13{\cdot}6-\frac{1}{16}\right]=V'\left[\frac{15}{16}\times 13{\cdot}6+\frac{1}{16}-10{\cdot}5\right].$$

$$\therefore\ V[308-204-1]=V'[204+1-168],\ \textit{i.e.}\ 103V=37V'.$$

Therefore required ratio

$$\frac{V\times 19{\cdot}25}{V'\times 10{\cdot}5}=\frac{37}{103}\times\frac{77}{42}=\frac{407}{618}.$$

5. Original depth in water $=\frac{9}{10}\times 40$ cms. $=36$ cms.

If x be the final depth in water, then

$$x\times 1+(40-x)\times{\cdot}6=40\times{\cdot}9.$$

$$\therefore\ x\times{\cdot}4=40\times{\cdot}3.\quad\therefore\ x=30.$$

Hence the wood rises 6 cms.

6. Originally the weight of the displaced air helped to support the body. When the air is removed, more liquid must be displaced and hence the body sinks.

7. We have $h\times n\rho=\frac{h}{2}\times\rho+\frac{h}{2}\times m\rho.$

$$\therefore\ n=\frac{m+1}{2}.$$

8. The first condition gives sp. gr. of the body $=\frac{1}{2}$.

In the second case if x be the fraction of the body in the water,

then $$1\times\frac{1}{2}=x\times 1+(1-x)\times 80\times{\cdot}00125=x+(1-x)\times{\cdot}1.$$

$$\therefore\ {\cdot}4=x\times{\cdot}9,\ \textit{i.e.}\ x=\frac{4}{9}.$$

9. Sp. gr. of cube $=\frac{4}{5}$.

Let x be the new fraction immersed; then

$$x\times 1+(1-x)\times \cdot 013=1\times\frac{4}{5}=\cdot 8.$$

$$\therefore\ x\times\cdot 987=\cdot 787.$$

$$\therefore\ x=\frac{787}{987}.$$

$$\therefore\ \frac{\text{new fraction}}{\text{old fraction}}=x\div\frac{4}{5}=\frac{5x}{4}=\frac{3935}{3948}.$$

10. Let x be the required depth of immersion, so that in the upper liquid there is $\frac{h}{n}-x$ and in the lower $h-\frac{h}{n}+x$. Hence we have

$$h\times\rho=\left(\frac{h}{n}-x\right)\rho_1+\left(h-\frac{h}{n}+x\right)\rho_2$$

$$=h\rho_2+\left(\frac{h}{n}-x\right)(\rho_1-\rho_2).$$

$$\therefore\ x=\frac{h}{n}-h\frac{\rho_2-\rho}{\rho_2-\rho_1}.$$

Clearly $\rho<\rho_2$; for otherwise the cylinder will entirely sink into the lower fluid. Also x must be positive;

$$\therefore\ \frac{\rho_2-\rho_1}{n}-\rho_2+\rho \text{ must be positive.}$$

$$\therefore\ \rho>\rho_2-\frac{\rho_2-\rho_1}{n}.$$

11. Let the length of the axis be h, and the length cut off be xh from the lower end of the axis. Then, since the volumes of similar cones vary as the cubes of their heights,

$$h^3\rho=x^3h^3\,\sigma_1+(h^3-x^3h^3)\,\sigma_2.$$

$$\therefore\ x^3=\frac{\rho-\sigma_2}{\sigma_1-\sigma_2}.$$

EXAMPLES. XII. (Pages 75—77.)

1. Tension = wt. of body − wt. of liquid displaced.

In the first case it $=18-\frac{18}{3}=12$ lbs. wt.

In the second it $=18-\frac{18\times 2}{3}=6$ lbs. wt.

2. $T = \text{wt. of platinum} - \frac{5}{24} \times \frac{\text{wt. of platinum}}{21}$

$$- \frac{19}{24} \times \frac{\text{wt. of platinum}}{21} \times 13$$

$$= \text{wt. of platinum} \times \left[1 - \frac{5+247}{504}\right]$$

$$= \frac{1}{2} \text{ wt. of platinum.}$$

3. Let V_1, V_2 be their volumes so that

$$V_1 [10{\cdot}5 - {\cdot}85]\, w = V_2 [19{\cdot}3 - 1{\cdot}5],$$

i.e. $$V_1 \times 9{\cdot}65 = V_2 \times 17{\cdot}8.$$

Therefore ratio required

$$= \frac{V_1 \times 10{\cdot}5}{V_2 \times 19{\cdot}3} = \frac{105}{193} \times \frac{1780}{965} = \frac{37380}{37249}.$$

4. Apparent weight

$$= 1 \text{ cwt.} - \frac{1 \text{ cwt.}}{7{\cdot}6} = \frac{66}{76} \text{ cwt.} = 97\tfrac{5}{19} \text{ lbs. wt.}$$

Let x be the required number of lbs. of wood. Then

$$112 + x = \frac{112}{7{\cdot}6} + \frac{x}{{\cdot}6}.$$

$$\therefore 112 \times \frac{66}{76} = x \times \frac{2}{3}.$$

$$\therefore x = \frac{3}{2} \times \frac{66}{76} \times 112 = 145\tfrac{17}{19} \text{ lbs. wt.}$$

5. If σ be the sp. gr., then

$$1 - \frac{1}{\sigma} = \frac{35}{37}.$$

$$\therefore \sigma = 18{\cdot}5.$$

6. Thrust $=$ wt. of body $-$ wt. of displaced water

$= (30 \times 1{\cdot}5 - 30 \times 1)$ grammes wt. $= 15$ grammes wt.

7. V_1 and V_2 being the volumes, we have $(V_1 + V_2) \times w =$ wt. of the water displaced

$$= \frac{1}{14} [V_1 \times 19{\cdot}25 + V_2 \times 10{\cdot}5] \,.\, w.$$

$$\therefore V_1 [19{\cdot}25 - 14] = V_2 [14 - 10{\cdot}5].$$

$$\therefore \frac{V_2}{V_1} = \frac{5{\cdot}25}{3{\cdot}5} = \frac{3}{2}.$$

8. wt. of lead – wt. of water displaced by the lead = wt. of wood – wt. of water displaced by the wood.

$\therefore$ wt. of wood – wt. of lead
= wt. of water displaced by the wood
– wt. of water displaced by the lead
= a positive quantity clearly. $\therefore$ etc.

9. Let V_1, V_2 be the volumes and σ_1, σ_2 the densities. Then

$$V_1\sigma_1 = 2 \cdot V_2\sigma_2 \text{ and } V_1(\sigma_1 - 1) = V_2(\sigma_2 - 1).$$

$$\therefore \frac{\sigma_1}{\sigma_1 - 1} = \frac{2\sigma_2}{\sigma_2 - 1}.$$

$$\therefore \frac{5}{2} = \frac{2\sigma_2}{\sigma_2 - 1}, \quad \therefore \sigma_2 = 5.$$

11. In the first case there is no change since the displaced water runs over the side of the vessel.

In the second case the thrust on the base is increased by the weight of the water displaced by the metal.

12. The sp. gr. of the wood is $\frac{2}{3}$. Let V be the required volume.

Then
$$\left(26 \times \frac{2}{3} + V \times 8 \times \frac{2}{3}\right) w = (26 + V) \cdot w.$$

$$\therefore V\left[\frac{16}{3} - 1\right] = 26 - 26 \times \frac{2}{3} = 26 \times \frac{1}{3}.$$

$$\therefore V = 2 \text{ cub. ins.}$$

The upward force required = half the wt. of the body

$$= \frac{1}{2}\left[26 \times \frac{2}{3} + 2 \times 8 \times \frac{2}{3}\right] \times w$$

$$= \frac{42}{3} \times \frac{1000}{1728} \text{ ozs. wt.} = \frac{14000}{1728} \text{ ozs. wt.} = \frac{875}{1728} \text{ lbs. wt.}$$

13. In the first case the thrust is increased by the weight of the displaced water.

The volume of 56 lbs. of iron $= \frac{56}{440}$ cub. ft.

Therefore the weight of the displaced water $= \frac{56}{440} \times 62\frac{1}{2}$ lbs. $= 7\frac{21}{22}$ lbs.

In the second case the thrust is increased by the weight of the iron.

14. If σ be the sp. gr. of the body, then

$$W' = W - \frac{W}{\sigma} = W\left(1 - \frac{1}{\sigma}\right) \qquad \text{.....................(1).}$$

Its weight in air

$$= W - W\frac{s}{\sigma} = W - s[W - W'].$$

15. Let W_1 be the true weight, and σ the sp. gr. of the body, so that

$$W = W_1 - W_1\frac{s}{\sigma},$$

$$W' = W_1 - W_1 \cdot \frac{1}{\sigma}.$$

$$\therefore W - sW' = W_1 - sW_1.$$

$$\therefore W_1 = \frac{W(1-s) + s(W - W')}{1-s} = W + \frac{s}{1-s}(W - W').$$

16. Let W be the true weight and σ the sp. gr. of the body. Then

$$w_1 = W - W\frac{s_1}{\sigma},$$

$$w_2 = W - W\frac{s_2}{\sigma},$$

and

$$w_3 = W - W\frac{s_3}{\sigma}.$$

$$\therefore w_1(s_2 - s_3) + w_2(s_3 - s_1) + w_3(s_1 - s_2) = 0.$$

17. Let the sp. grs. of the liquids be s_1, s_2, s_3; let the true weights of the solids be W' and W'' and their sp. grs. s' and s''. Then

$$w_1 = W'\left[1 - \frac{s_1}{s'}\right];\ w_2 = W'\left[1 - \frac{s_2}{s'}\right];\ w_3 = W'\left[1 - \frac{s_3}{s'}\right];$$

$$W_1 = W''\left[1 - \frac{s_1}{s''}\right];\ W_2 = W''\left[1 - \frac{s_2}{s''}\right];\ W_3 = W''\left[1 - \frac{s_3}{s''}\right].$$

$$\therefore w_1[W_2 - W_3] + \ldots + \ldots$$

$$= W'W''\left[1 - \frac{s_1}{s'}\right]\left[\frac{s_3 - s_2}{s''}\right] + \ldots + \ldots$$

$$= \frac{W'W''}{s's''}[s'(s_3 - s_2) + \ldots + \ldots - s_1(s_3 - s_2) - \ldots - \ldots] = 0.$$

EXAMPLES. XIII. (Pages 78, 79.)

1. Tension of the string = wt. of displaced water − wt. of the cork

$$=\left(\frac{30}{\cdot 25}-30\right) \text{ grammes wt.} = 90 \text{ grammes wt.}$$

2. The volume of the wood $=\frac{6}{\cdot 8\times w}$ cub. ft. $=\frac{15}{2w}$, and the weight of this volume of the mixture must = 8 lbs. when the string is just about to break. Hence, if x be the final proportion of the whole barrel which consists of the liquid of sp. gr. 1·2, we have

$$8=\frac{15}{2w}\times\left\{\frac{(1-x)\times 1+x\times 1\cdot 2}{1-x+x}\right\}\times w$$

$$=\frac{15}{2}(1+\cdot 2\times x)=\frac{15}{2}+\frac{3}{2}x.$$

$$\therefore\ x=\frac{1}{3}.$$

If a greater proportion than this be of the higher sp. gr. the upward thrust of the displaced fluid would be too great and the string would break.

3. Force = upward thrust due to the weight of the additional water displaced.

Now 15 lbs. = wt. of cylinder of water of length 18 ins.

Therefore force required $=\frac{6}{18}\times 15$ lbs. wt. = 5 lbs. wt.

4. Total upward thrust on balloon

= wt. of air displaced by the coal-gas

− wt. of the coal-gas

= 4000000 × [1·29 − ·52] grammes

= 3080000 grammes.

∴ additional wt. = (3080000 − 1500000) grammes

= 1580000 grammes wt.

5. Tension of the string

$$=10\times 1\cdot 25\times\left(1-\frac{1}{14\cdot 6}\right) \text{ ozs.}$$

$$=12\cdot 5\times\frac{136}{146} \text{ ozs.} = 11\tfrac{47}{73} \text{ ozs.}$$

6. Upward thrust = wt. of displaced air − wt. of balloon

$$=\left(64000\times\frac{1\cdot 24}{16}-4480\right)\text{ lbs. wt.}=480\text{ lbs. wt.}$$

$$\therefore\ \text{acceleration}=\frac{480g}{4480}=\frac{3g}{28}.$$

7. Draw AD perpendicular to BC; bisect AC, CD in K, L. Then KL is the water-line and the area

$$CKL=\frac{1}{4}\text{ area } CAD\ [\text{Euc. VI. 19}]=\frac{1}{8}\text{ area } ACB.$$

If σ be the sp. gr. of the lamina, the weight of the volume

$$AKLB\text{ of water}=\frac{7}{8}\frac{W}{\sigma}.$$

Hence the tension of the string

$$=\frac{7W}{8\sigma}-W\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again the upward thrust of the water

= difference between the triangles ABC, KLC

$=\dfrac{W}{\sigma}$ acting at C.G. of $\triangle ABC$

$-\dfrac{1}{8}\dfrac{W}{\sigma}$ acting at C.G. of $\triangle KLC$.

Hence taking moments about A we have

$$W\times\frac{2}{3}AD=\frac{W}{\sigma}\times\frac{2}{3}AD-\frac{1}{8}\frac{W}{\sigma}\left[\frac{1}{2}AD+\frac{2}{3}KL\right]$$

$$=\frac{2W.AD}{3\sigma}-\frac{W}{8\sigma}\times\frac{5}{3}\times\frac{AD}{2}.$$

$$\therefore\ 1=\frac{1}{\sigma}-\frac{5}{32\sigma}=\frac{27}{32\sigma}.$$

$$\therefore\ \sigma=\frac{27}{32},$$

and therefore (1) gives

$$\text{tension}=\frac{7}{8}\times\frac{32}{27}.W-W=\frac{W}{27}.$$

EXAMPLES. XIV. (Page 81.)

1. If σ be the sp. gr. of the rod, and W be its weight, the weight of the displaced fluid is $\frac{2}{3}\frac{W}{\sigma}$. Hence, taking moments about the fulcrum, we have

$$W\times 3=\frac{2}{3}\cdot\frac{W}{\sigma}[2+2]=\frac{8}{3}\cdot\frac{W}{\sigma}.$$

$$\therefore\ \sigma=\frac{8}{9}.$$

2. If W be the wt. of the rod, the wt. of the displaced water

$$=\frac{1}{2}\times\frac{W}{2{\cdot}5}=\frac{W}{5}.$$

If T and T' be the tensions of the strings attached to the upper and lower ends, we have

$$T+T'=W-\frac{W}{5}=\frac{4W}{5}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, by taking moments about the lower end, we have

$$T\times 1+\frac{W}{5}\times\frac{1}{4}=W\times\frac{1}{2}.$$

$$\therefore\ T=\frac{W}{2}-\frac{W}{20}=\frac{9W}{20}.$$

Also $$T'=\frac{4W}{5}-\frac{9W}{20}=\frac{7W}{20}.\quad\therefore\ \frac{T}{T'}=\frac{9}{7}.$$

3. If W be the weight of the rod and σ its sp. gr. the weight of the water displaced $=\frac{1}{3}\cdot\frac{W}{\sigma}$, and the points of action of these forces are at a distance from the fulcrum equal to $\frac{1}{2}$ and $\left(\frac{2}{3}+\frac{1}{6}\right)$ of its length. Hence, on taking moments,

$$W\times\frac{1}{2}=\frac{1}{3}\frac{W}{\sigma}\times\left(\frac{2}{3}+\frac{1}{6}\right)=\frac{W}{3\sigma}\times\frac{5}{6}.$$

$$\therefore\ \sigma=\frac{5}{9}.$$

4. The length of the rod out of the liquid $=\frac{h}{\cos\theta}$, and hence the weight of the displaced fluid

$$=\frac{W\,.\,\sigma}{\rho}\,\frac{2a-\dfrac{h}{\cos\theta}}{2a}.$$

Hence, by taking moments,

$$W \times a = \frac{W.\sigma}{\rho}\left[1 - \frac{h}{2a\cos\theta}\right]\left[\frac{h}{\cos\theta} + \frac{1}{2}\left(2a - \frac{h}{\cos\theta}\right)\right].$$

$$\therefore \frac{\rho}{\sigma} = \left[1 - \frac{h}{2a\cos\theta}\right]\left[1 + \frac{h}{2a\cos\theta}\right] = 1 - \frac{h^2}{4a^2\cos^2\theta}.$$

$$\therefore \frac{h^2}{4a^2\cos^2\theta} = 1 - \frac{\rho}{\sigma} = \frac{\sigma - \rho}{\sigma}.$$

$$\therefore \cos\theta = \frac{h}{2a}\sqrt{\frac{\sigma}{\sigma - \rho}}.$$

Also the rod could clearly rest in a vertical position; but this latter position of equilibrium would be found to be unstable.

EXAMPLES. XV. (Page 84.)

1. The metacentre is at the centre of the ball. If a weight be placed at the highest point, the resulting centre of gravity is now above the centre, *i.e.* above the metacentre. Hence the equilibrium is now unstable.

2. The metacentre is at the centre of the common base of the cone and hemisphere. The centre of gravity must therefore not be above the base. Hence, if h be the height and r the radius of the base of the cone, we have

$$\frac{1}{3}\pi r^2 h \times \frac{h}{4} \leqq \frac{2}{3}\pi r^3 \times \frac{3r}{8}.$$

$$\therefore h \leqq \sqrt{3}r.$$

3. Similarly in this case since the bodies are hollow we have

$$\frac{1}{2}.2\pi r.l \times \frac{h}{3} \leqq 2\pi r^2 \times \frac{r}{2}.$$

$$\therefore 1 \leqq 3\frac{r^2}{lh} \leqq 3\sin\alpha\tan\alpha,$$

i.e.

$$3\sin^2\alpha \geqq \cos\alpha.$$

This condition is clearly satisfied when $\alpha = 45°$, but not when $\alpha = 30°$.

4. As in the previous examples the metacentre is at the centre of the base of the hemisphere. Hence the centre of gravity must not be above this centre.

Hence, if h be the height and r the radius of the base, then

(1) $$\pi r^2 h \times \frac{h}{2} \leqq \frac{2}{3}\pi r^3 \times \frac{3r}{8}.$$

$$\therefore\ h^2 \leqq \frac{r^2}{2},$$

i.e. $$h \leqq \frac{1}{2} \cdot r\sqrt{2}.$$

(2) $$2\pi r h \times \frac{h}{2} \leqq 2\pi r^2 \times \frac{r}{2},$$

i.e. $$h^2 \leqq r^2,$$

i.e. $$h \leqq r.$$

EXAMPLES. XVI. (Pages 87—90.)

1. Let the base be inclined at θ to the horizon when there is equilibrium. The pressure of the water at each point of the surface in contact with it acts through the centre of the hemisphere; the resultant thrust of the water therefore goes through the centre. Hence the moments of the w and W about the centre must balance.

$$\therefore\ W \times \frac{3a}{8}\sin\theta = w \times a\cos\theta.$$

$$\therefore\ \tan\theta = \frac{8w}{3W}.$$

2. Since it floats however placed the C.G. of the whole surface of the cone must coincide with that of the fluid displaced.

$$\therefore\ \frac{\frac{1}{2}\cdot 2\pi r l \cdot \frac{h}{3} + \pi r^2 \cdot 0}{\frac{1}{2}\cdot 2\pi r l \cdot + \pi r^2} = \frac{h}{4}.$$

$$\therefore\ l = 3r,\ \textit{i.e.}\ \text{vertical} \angle = 2\sin^{-1}\frac{1}{3}.$$

3. The pressures caused by the bar upon the two parallelopipeds are respectively 75 and 25 lbs. wt.

The total weight to press the larger one down thus

$$= 175 = \frac{7}{8} \times \text{wt. water displaced by it.}$$

$$\therefore\ \frac{1}{8}\ \text{of it is above the water.}$$

Similarly the weight to press the smaller one down

$$=75 \text{ lbs.} = \frac{3}{4} \times \text{wt. water displaced by it.}$$

$$\therefore \frac{1}{4} \text{ of it is above the water.}$$

4. Let the section of the prism be the right-angled triangle ABC, having $AB = AC$, a right angle at A and the point B in the water. Draw BD a horizontal line through B to meet AC in D. Then, since the sp. gr. of the body is $\frac{1}{2}$,

$$\therefore \triangle ABD = \frac{1}{2} \triangle ABC.$$

$$\therefore AD = \frac{1}{2} AC.$$

$$\therefore \tan ABD = \frac{AD}{AB} = \frac{1}{2}.$$

$$\therefore \tan CBD = \tan(45^\circ - ABD) = \frac{1 - \frac{1}{2}}{1 + \frac{1}{2}} = \frac{1}{3}.$$

5. Let the wedge float with the edge A immersed. Through C draw CK horizontal to meet AB in K.

Then the wt. of the wedge may be replaced by three equal weights, W, at A, C, B, and similarly the thrust of the water by upward forces, each equal to W, at A, K, C. These cannot balance unless the forces at K and B balance, *i.e.* unless BK, and therefore AB, is vertical.

We then have, if σ be the sp. gr. of the wedge,

$$\triangle ACK \,.\, w = \sigma \times \triangle ACB \times w.$$

$$\therefore \sigma = \frac{\triangle ACK}{\triangle ACB} = \frac{AK}{AB} = \frac{AK}{AC}\frac{AC}{AB} = \frac{\sin B \,.\, \cos A}{\sin C}.$$

Similarly if it float with the vertex B downward the sp. gr. must be

$$\frac{\sin A \cos B}{\sin C}.$$

6. Let V be the vertex, VO the axis of the cone; A the point of the base on the surface of the water and AOB the diameter of the base through A. Draw VD perpendicular to the surface of the water and let this surface cut VB in C.

Let $$\angle AVD = \theta.$$

For equilibrium the C.G. of the fluid displaced (viz. the cone VCA) must be vertically under that of the cone VAB.

The horizontal distance from V of the first

$$=\frac{3}{4}\left[\frac{DC+DA}{2}\right]=\frac{3}{8}\,.\,VD\,[\tan(\theta-60^\circ)+\tan\theta]$$

$$=\frac{3}{8}\,.\,VA\cos\theta\times\frac{\sin(2\theta-60^\circ)}{\cos\theta\cos(\theta-60^\circ)}\text{ .... (1).}$$

The horizontal distance of the former

$$=\frac{2}{3}\,VO\sin(\theta-30^\circ)=\frac{2}{3}\,.\,VA\cos 30^\circ\,.\,\sin\overline{\theta-30^\circ}\text{ .... (2).}$$

Equating (1) and (2), we have

$$\frac{3}{8}\times\frac{2\sin\overline{\theta-30^\circ}\cos\overline{\theta-30^\circ}}{\cos(\theta-60^\circ)}=\sin(\theta-30^\circ)\times\frac{\sqrt{3}}{3}.$$

$$\therefore\ 9\left[\cos\theta\frac{\sqrt{3}}{2}+\sin\theta\,.\,\frac{1}{2}\right]=4\sqrt{3}\cos(\theta-60^\circ)$$

$$=2\sqrt{3}\cos\theta+6\sin\theta.$$

$$\therefore\ \tan\theta=\frac{5\sqrt{3}}{3}=\frac{5}{\sqrt{3}}.$$

$$\therefore\ \tan(\angle\text{ with horizontal})=\cot\theta=\frac{\sqrt{3}}{5}.$$

7. Let AOB be the section of the base of the hemisphere by the plane of the paper, A being the highest point and O the centre, and let G be the C.G. of the bowl. Then the weight of the bowl will clearly have the greatest effect when OG is most nearly horizontal, *i.e.* when AB is most nearly vertical, *i.e.* when the surface of the water passes through the lowest point B, and then ϕ is the angle that OB makes with the horizontal or OG with the vertical. Taking moments about the point of contact, C, of the bowl and plane, we then have

$$\text{wt. of water}\times OC\sin\alpha=\text{wt. of bowl}\times\left[\begin{matrix}OG\sin\phi\\-OC\sin\alpha\end{matrix}\right],$$

since the wt. of the water must pass through O.

$$\therefore\ \frac{\text{wt. of bowl}}{\text{wt. of water}}=\frac{a\sin\alpha}{\frac{a}{2}\sin\phi-a\sin\alpha}=\frac{2\sin\alpha}{\sin\phi-2\sin\alpha}.$$

8. Let W' be the weight of the ball; the portion $\frac{W'}{\sigma}$ of its weight is supported by the water and the tension of the string is thus

$W' - \frac{W'}{\sigma}$. But this $\frac{W'}{\sigma}$ is added to the weight of the water so that the tension at the other end of the string $= W + \frac{W'}{\sigma}$. Equating these, we have

$$W = W' - \frac{2W'}{\sigma} = W'\frac{\sigma - 2}{\sigma}, \text{ i.e. } W' = \frac{\sigma W}{\sigma - 2}.$$

This is the weight of the ball when there is equilibrium with it entirely immersed in the water. There is also equilibrium when it is entirely unimmersed if the weight of the ball be W. There will be equilibrium for any value of W' between W and $\frac{\sigma W}{\sigma - 2}$, a corresponding part of the ball being immersed.

9. The acceleration of either bucket

$$= \frac{m - m'}{2M + m + m'} g = f.$$

The upward thrust of the water on the mass m being P and the tension of the string T, we have

$$T + mg - P = mf \text{ .... (1).}$$

If the mass m were removed and replaced by a mass $\frac{m}{\sigma}$ of water this thrust P together with its weight would give it the acceleration f.

$$\therefore \frac{m}{\sigma} g - P = \frac{m}{\sigma} f \text{ .... (2).}$$

Subtracting (2) from (1), we have

$$T = m(f - g)\left(1 - \frac{1}{\sigma}\right)$$

$$= m\left(\frac{1}{\sigma} - 1\right)\frac{2M + 2m'}{2M + m + m'}.$$

10. x being the length immersed initially, we have

$$x\sigma_1 + (h - x)\sigma_2 = h\rho.$$

x' being the final length, we have

$$x'\sigma_1 + (h - x')\sigma_3 = h\rho.$$

$$\therefore x - x' = \frac{h(\rho - \sigma_2)}{\sigma_1 - \sigma_2} - \frac{h(\rho - \sigma_3)}{\sigma_1 - \sigma_3} = h\frac{(\sigma_1 - \rho)(\sigma_3 - \sigma_2)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)},$$

i.e. it rises through this distance.

11. As in the last example,

$$x = h\,\frac{\rho - \sigma_2}{\sigma_1 - \sigma_2}.$$

$$\therefore\; x - h = h\,\frac{\rho - \sigma_1}{\sigma_1 - \sigma_2}.$$

Therefore if σ_2 be changed to $\sigma_2 + \lambda$ where λ is small, and in consequence x to $x - \xi$ where ξ is small, then

$$\xi = h\,\frac{\rho - \sigma_1}{\sigma_1 - \sigma_2} - h\,\frac{\rho - \sigma_1}{\sigma_1 - \sigma_2 - \lambda}$$

$$= \frac{h\lambda\,(\sigma_1 - \rho)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_2 - \lambda)}$$

$$= \frac{h\lambda\,(\sigma_1 - \rho)}{(\sigma_1 - \sigma_2)^2} \text{ nearly}$$

$$= (h - x)^2 \cdot \frac{\lambda}{h\,(\sigma_1 - \rho)},$$

i.e. $\xi \propto (h - x)^2$

$\propto$ sq. of unimmersed portion.

12. Let x be the length of the cork which is immersed, so that

$$x + (h - x)\,\sigma = hs,$$

i.e.
$$x = \frac{h\,(s - \sigma)}{1 - \sigma}.$$

When the air is pumped out let x become x_1;

$$\therefore\; x_1 = hs.$$

Therefore distance through which the cork sinks

$$= x_1 - x = hs - \frac{h\,(s - \sigma)}{1 - \sigma} = \frac{h\sigma\,(1 - s)}{1 - \sigma}.$$

13. Let V be the total volume of the body and σ its density, so that

$$(V - P_1) \cdot 1 + P_1 \cdot \rho_1 = V \cdot \sigma.$$

$$\therefore\; P_1 = V\,\frac{1 - \sigma}{1 - \rho_1}.$$

$$\therefore\; \frac{V}{P_1} = \frac{1 - \rho_1}{1 - \sigma}; \text{ so } \frac{V}{P_2} = \frac{1 - \rho_2}{1 - \sigma} \text{ and } \frac{V}{P_3} = \frac{1 - \rho_3}{1 - \sigma}.$$

$$\therefore\; \frac{\rho_2 - \rho_3}{P_1} + \frac{\rho_3 - \rho_1}{P_2} + \frac{\rho_1 - \rho_2}{P_3} = 0.$$

14. Let the volumes be V_1 and V_2.

$$\therefore\ (V_1\sigma_1+V_2\sigma_2)\,w=a \ldots\ldots\ldots\ldots(1),$$

$$[V_1(\sigma_1-1)+V_2(\sigma_2-1)]\,w=b,$$

i.e.
$$(V_1+V_2)\,w=a-b \ldots\ldots\ldots\ldots(3).$$

From (1) and (3)

$$[V_1\sigma_1+V_2\sigma_2][a-b]=(V_1+V_2)\times a.$$

$$\therefore\ V_1[(a-b)\,\sigma_1-a]=V_2[a-\sigma_2(a-b)].$$

15. Let W and W' be the apparent weights, and w the real weight, so that

$$W\left[1-\frac{\sigma}{\rho'}\right]=w\left[1-\frac{\sigma}{\rho}\right],$$

and
$$W'\left[1-\frac{\sigma'}{\rho'}\right]=w\left[1-\frac{\sigma'}{\rho}\right].$$

$$\therefore\ \frac{W-W'}{W}=1-\frac{\rho-\sigma'}{\rho'-\sigma'}\,\frac{\rho'-\sigma}{\rho-\sigma}=\frac{(\sigma'-\sigma)\,(\rho'-\rho)}{(\rho'-\sigma')\,(\rho-\sigma)}=\text{a positive quantity},$$

since $\rho'>\rho$, $\sigma'>\sigma$, and $\rho>\sigma$, $\rho'>\sigma'$ clearly.

16. Let $\sigma=\cdot00125$ and $\rho=8\cdot4$ and let the true weight of the water be W and the true weight of the "weights" W'. Then

$$W'\left[1-\frac{\sigma}{\rho}\right]$$

$=$ tension of the string which supports W'

$=$ " " " " " W

$=W[1-\sigma]$.

The result required

$$=\frac{W-W'}{W'}=\frac{W}{W'}-1$$

$$=\frac{1-\dfrac{\sigma}{\rho}}{1-\sigma}-1=\frac{\sigma\left[1-\dfrac{1}{\rho}\right]}{1-\sigma}=\sigma\left[1-\frac{1}{\rho}\right],\ \text{since } \sigma \text{ is very small},$$

$$=\cdot00125\times\frac{7\cdot4}{8\cdot4}=\frac{1}{800}\times\frac{37}{42}=\frac{1}{100}\times\frac{37}{336}$$

$$=\frac{\cdot11\ldots}{100}=\text{about } \cdot1 \text{ per cent.}$$

17. Let W be the weight of each ball, and x, y, z their specific gravities.

Then $$W\left[1-\frac{\sigma_1}{x}\right]=W\left[1-\frac{\sigma_1}{y}\right]+W\left[1-\frac{\sigma_1}{z}\right].$$

$$\therefore \frac{1}{y}+\frac{1}{z}-\frac{1}{x}=\frac{1}{\sigma_1}.$$

So $$\frac{1}{z}+\frac{1}{x}-\frac{1}{y}=\frac{1}{\sigma_2},$$

and $$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=\frac{1}{\sigma_3}.$$

$$\therefore \frac{2}{x}=\frac{1}{\sigma_2}+\frac{1}{\sigma_3}, \text{ so that } x=\frac{2\sigma_2\sigma_3}{\sigma_2+\sigma_3}.$$

So y and z.

18. Let the surface of the liquid cut CA, CB in D and E, so that

$$CE=\frac{1}{2}CB=\frac{a}{2},$$

and $$CD=CE\tan 60^\circ=\sqrt{3}\,.\frac{a}{2}.$$

$$\therefore \frac{\triangle CDE}{\triangle CAB}=\frac{CD\,.\,CE}{CA\,.\,CB}=\frac{\sqrt{3}}{4}\frac{a}{b}.$$

If the weight of the $\triangle CAB$ be $3W$ it may be replaced by W at each of the angular points A, B, C; also the weight of the $\triangle CDE$ of liquid will be $\frac{8}{3}.3W\times\frac{\sqrt{3}}{4}\frac{a}{b}$ and it may thus be replaced by $\frac{2\sqrt{3}}{3}\frac{a}{b}W$ at each of C, D, E.

Taking moments about C, we have

$$W.CA\cos 30^\circ-\frac{2\sqrt{3}}{3}\frac{a}{b}W.CD\cos 30^\circ$$

$$=W.CB\cos 60^\circ-\frac{2\sqrt{3}}{3}.\frac{a}{b}.W.\frac{a}{2}\cos 60^\circ,$$

i.e. $$b\frac{\sqrt{3}}{2}-\frac{2\sqrt{3}}{3}.\frac{a}{b}.\frac{a}{2}\tan 60^\circ\cos 30^\circ$$

$$=a.\frac{1}{2}-\frac{2\sqrt{3}}{3}.\frac{a}{b}.\frac{a}{2}.\frac{1}{2}.$$

$$\therefore 3b\sqrt{3}=2\frac{a^2}{b}\sqrt{3}+3a,$$

i.e. $$2a^2+\sqrt{3}ab-3b^2=0.$$

$$\therefore\ (2a-\sqrt{3}b)(a+\sqrt{3}b)=0.$$

$$\therefore\ 2a=\sqrt{3}b,$$

i.e. $$\frac{CA}{CB}=\frac{b}{a}=\frac{2}{\sqrt{3}}.$$

19. Let $ABDC$ be the lamina, the fixed point being E, the middle point of the shorter side AC; let CB be the diagonal in the surface of the liquid and A the vertex under the liquid. Let $AC=a$, $AB=a\sqrt{3}$ and $\therefore\ \angle ABC=30°$. Let ρ be sp. gr. of lamina, σ that of the liquid.

Then wt. of lamina $=a^2\sqrt{3}\rho$ and acts at O, the middle point of BC.

The wt. of the liquid $=\frac{1}{2}a^2\sqrt{3}\sigma$, and may be replaced by $\frac{a^2\sqrt{3}\sigma}{6}$ acting upwards at A, B, C.

Taking moments about E, we have

$$a^2\sqrt{3}\rho\times\frac{\sqrt{3}a}{2}\cos 30°$$

$$=\frac{a^2\sqrt{3}\sigma}{6}\left[\frac{a}{2}\cos 60°+\left(\frac{a}{2}\cos 60°+\sqrt{3}a\cos 30°\right)-\frac{a}{2}\cos 60°\right]$$

$$=\frac{a^3\sqrt{3}\cdot\sigma}{6}\left[\frac{1}{2}\cos 60°+\sqrt{3}\cos 30°\right],$$

i.e. $$\frac{3a^3\rho}{2}\cdot\frac{\sqrt{3}}{2}=\frac{a^3\sqrt{3}\sigma}{6}\times\frac{7}{4}.$$

$$\therefore\ \frac{\rho}{\sigma}=\frac{4}{3}\cdot\frac{7}{24}=\frac{7}{18}.$$

Also downward thrust on E

$$=\frac{a^2\sqrt{3}\sigma}{2}-a^2\sqrt{3}\rho$$

$$=\frac{9}{7}a^2\sqrt{3}\rho-a^2\sqrt{3}\rho$$

$$=\frac{2}{7}a^2\sqrt{3}\rho=\frac{2}{7}\times\text{wt. of lamina.}$$

20. Let the surface of the water through B cut CD in E, and let $\angle ABE=\beta$, so that $\sin\beta=\frac{4}{5}$, and hence $\cos\beta=\frac{3}{5}$, and

$$CE=CB\cot\beta=5\times\frac{3}{4}=\frac{15}{4}.$$

$$\therefore\ \text{area of }\triangle ECB=\frac{1}{2}\cdot EC\cdot CB=\frac{75}{8}.$$

Now the upward thrust of the water is $w \times$ the area of $DEBA$, and is thus equal to $w \times$ area $ABCD$ acting upward at O the middle point of AC, and to $w \times$ area ECB acting downwards at its C.G. The latter is equal to $\frac{25}{8}w$ at each of the vertices E, B, C.

Also the weight of the square is $25\rho w$ at O. Hence taking moments about A, we have

$$25(\rho-1)w \times \frac{AC}{2}\cos(\beta+45^\circ)+\frac{25}{8}w\,[AB\cos\beta+(AB\cos\beta-BC\sin\beta)$$
$$-(AD\sin\beta-DE\cos\beta)]=0,$$

i.e. $$8(1-\rho)\frac{5\sqrt{2}}{2}\left[\frac{\cos\beta-\sin\beta}{\sqrt{2}}\right]=10\cos\beta-10\sin\beta+\frac{5}{4}\cos\beta.$$

$$\therefore\ 4(1-\rho)(1-\tan\beta)=2-2\tan\beta+\frac{1}{4}=\frac{9-8\tan\beta}{4}.$$

$$\therefore\ (1-\rho)=\frac{1}{16}\frac{8\tan\beta-9}{\tan\beta-1}=\frac{1}{16}\frac{32-27}{4-3}=\frac{5}{16}.$$

$$\therefore\ \rho=\frac{11}{16}.$$

21. Let $ABCD$ be the rectangle, A being outside the liquid; let AB $(=a)$ be less than AD $(=b)$.

Let $\alpha=\angle DAC$, so that

$$\frac{\sin\alpha}{a}=\frac{\cos\alpha}{b}=\frac{1}{\sqrt{a^2+b^2}}.$$

Since half the area is in the liquid the middle point, O, of the diagonal AC must be in the surface of the liquid. Let the surface cut AD in E and CB in F. Draw EK parallel to DC to meet CB in K; draw AH perpendicular to EF.

Then

$$AE=\sqrt{2}\,.\,AH=\sqrt{2}\,.\,\frac{\sqrt{a^2+b^2}}{2}\sin(\alpha+45^\circ)=\frac{a+b}{2}.$$

$$\therefore\ DE=\frac{b-a}{2}=FB,\ \text{so that } KF=b-2DE=a.$$

$$\therefore\ \text{area } DEKC=\frac{a(b-a)}{2},$$

$$\text{and area } \triangle EKF=\frac{a^2}{2}.$$

Hence the forces acting are, $ab\rho w$ at O,

$$\frac{a(b-a)}{2}\sigma w \text{ at the middle point of } CE,$$

$$\frac{a^2}{6}\sigma w \text{ at each of } E, K, F,$$

where ρ, σ are the densities of the rectangle and liquid.

Hence, taking moments about A, we have

$$ab\rho \times AO\cos(\alpha+45°) = \frac{a^2\sigma}{6}\left[\frac{a+b}{2}\cos 45° + \left(\frac{a+b}{2}\cos 45° - a\cos 45°\right) - \left(a\cos 45° - \frac{b-a}{2}\cos 45°\right)\right]$$
$$+\frac{a(b-a)}{2}\sigma\left[\frac{1}{2}\left(b+\frac{b+a}{2}\right)\cos 45° - \frac{a}{2}\sin 45°\right],$$

i.e.
$$\frac{ab\rho}{2}\frac{b-a}{\sqrt{2}} = 3\frac{a^2\sigma}{6}\left[\frac{b-a}{2\sqrt{2}}\right] + \frac{a(b-a)}{2}\sigma \times \frac{3b-a}{4\sqrt{2}}.$$

$$\therefore \frac{ab\rho}{2} = \frac{a^2\sigma}{4} + \frac{a(3b-a)\sigma}{8}.$$

$$\therefore 4b\rho = 2a\sigma + (3b-a)\sigma.$$

$$\therefore \frac{\rho}{\sigma} = \frac{3b+a}{4b}.$$

22. Let $\angle DAC = \alpha$, so that

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}} \text{ and } \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}.$$

Let the water-surface cut DC in E and AB in F; draw EK parallel to AD to meet AB in K and let $AK = DE = x$.

Then $$\frac{c}{\sin\theta} = AF = FK + KA = 2a\cot\theta + x \quad\text{.................(1).}$$

Also

$$\text{area } \triangle KEF = \frac{1}{2}2a\cot\theta \,.\, 2a = 2a^2\cot\theta.$$

The forces acting are then

$4abw\sigma$ at O, the middle point of AC downwards,

$x \,.\, 2aw$ at the middle point of AE and upwards,

$\frac{2a^2}{3}\cot\theta \,.\, w$ upwards at each of the points K, E, F.

Hence, by taking moments about A, we have

$$4ab\sigma \cdot \frac{AC}{2} \cdot \sin \overline{\theta - \alpha}$$

$$= 2ax \times \frac{2a\sin\theta - x\cos\theta}{2} + \frac{2a^2}{3}\cot\theta \left[(2a\sin\theta - x\cos\theta) - x\cos\theta - \frac{c}{\sin\theta} \cdot \cos\theta \right].$$

$$\therefore\ 4b\sigma\,[a\sin\theta - b\cos\theta]$$

$$= x\,(2a\sin\theta - x\cos\theta) + \frac{2a}{3}\cot\theta\,[2a\sin\theta - 2x\cos\theta - c\cot\theta]$$

$$= \left(\frac{c}{\sin\theta} - 2a\cot\theta\right)\left[\frac{2a}{\sin\theta} - c\cot\theta\right]$$

$$+ \frac{2a}{3}\cot\theta\,[2a\sin\theta - 3c\cot\theta + 4a\cot\theta\cos\theta],$$

on substituting for x.

$$\therefore\ 12b\sigma\sin^2\theta\,(a\sin\theta - b\cos\theta)$$

$$= 3\,[c - 2a\cos\theta][2a - c\cos\theta] + 2a\cos\theta\,[2a\sin^2\theta - 3c\cos\theta + 4a\cos^2\theta]$$

$$= 6ac - 3c^2\cos\theta - 12a^2\cos\theta + 6ac\cos^2\theta$$

$$+ 4a^2\cos\theta\sin^2\theta - 6ac\cos^2\theta + 8a^2\cos^3\theta$$

$$= 6ac - 3c^2\cos\theta - 4a^2\cos\theta\,[3 - \sin^2\theta - 2\cos^2\theta]$$

$$= 6ac - 3c^2\cos\theta - 4a^2\cos\theta\,[2 - \cos^2\theta].$$

23. Let the angular point, A, of the square $ABCD$ be under the water, and let AB be inclined at θ to the horizontal.

Let the water line meet AD, AB in E and F and let $AE = x$ and hence $AF = x\cot\theta$.

Since the rectangle floats,

$$\therefore\ \frac{1}{2}x^2\cot\theta \cdot \sigma = a^2\rho \quad \text{.......................(1)},$$

where a is the side of the square.

If the weight of the rectangle be $3W$, then the upward thrust of the liquid is given by W at each of A, E, and F.

Taking moments about A, we have

$$3W \cdot \frac{\sqrt{2}\,a}{2} \cdot \cos(\theta + 45^\circ) = W\,[x\cot\theta\cos\theta - x\sin\theta].$$

$$\therefore\ \frac{3a}{2}(\cos\theta - \sin\theta) = x\,\frac{\cos^2\theta - \sin^2\theta}{\sin\theta}.$$

$\therefore\ 3a\sin\theta = 2x(\cos\theta + \sin\theta)$, or else $\cos\theta - \sin\theta = 0$, *i.e.* $\theta = 45^\circ$.

Squaring and using (1), we have

$$9a^2 \sin^2 \theta = 4x^2 (\cos \theta + \sin \theta)^2$$
$$= \frac{8a^2\rho}{\sigma} \tan \theta (\cos \theta + \sin \theta)^2.$$
$$\therefore 9\sigma \sin \theta \cos \theta = 8\rho (1 + 2 \sin \theta \cos \theta).$$
$$\therefore 9\sigma \sin 2\theta = 16\rho (1 + \sin 2\theta).$$
$$\therefore \sin 2\theta = \frac{16\rho}{9\sigma - 16\rho}.$$

Provided that $16\rho < 9\sigma - 16\rho$, *i.e.* that $32\rho < 9\sigma$, this gives two values of 2θ each less than 180° and hence two values of θ, less than 90°.

Also one value of θ has been found to be 45°, so that there are three positions of equilibrium.

24. Let a vertical section perpendicular to the horizontal axis of revolution cut the fixed cup in the semicircle *AKB*, *AOB* being a diameter; also let it cut the solid hemisphere in *CAKD*, *COD* being a diameter.

Draw *OK* below *OA* making $\angle KOA = \angle AOC$, and produce *KO* to meet in *L* the circle of which the fixed cup (produced) is a portion.

Clearly if the solid *CAKDBL* were uniform it would be in equilibrium, by symmetry.

Now the distance from *O* horizontally of the C.G. of the wedge *LOD* is clearly the same as that of *BOD*. Hence a wedge *BOD* of twice the density of *LOD* would have the same moment about *O* that *LOD* has.

Hence the moments about *O* of the solid wedge *CAKD*, and of the wedge *BOD* of twice the density, balance and thus there is equilibrium.

This holds whatever be the angle between *COD* and *AOB*.

Oil-lamps and inkstands have been constructed on this principle. *CAD* is a solid hemisphere turning round inside the container *AKB*. As the oil, or ink, is used the inclination of *CD* to *AB* is lessened, and the surface *OB* of the liquid always remains at the same level.

25. Let W be the counterpoise, A the area of the cross-section of the cylinder and l and σ its length and density, k the total length of the chain and w_1 its wt. per unit length, and let y be the length of the chain between the pulley and cylinder when a length x of the cylinder is in the water.

Then $$W + (k - y) w_1 = Al\sigma w - Axw + yw_1 \quad\text{.................(1).}$$

Also, if h be the height of the pulley above the water, then

$$h = y + l - x \quad\text{.................................(2).}$$

Substituting for y in (1), we have

$$W = Al\sigma w - Axw + w_1[2h + 2x - 2l - k].$$

This equation is true for all values of x, *i.e.* the cylinder will rest with any length immersed, if

$$W = Al\sigma w + w_1[2h - 2l - k],$$

and $$0 = -Aw + 2w_1,$$

i.e. if $$A = \frac{2w_1}{w},$$

and $$W = w_1[2h - k - 2l(1 - \sigma)].$$

26. Let O be the centre of the fixed sphere, O' that of the movable hemisphere, and A the point of contact. Let $2r$ and r be the radii of the spheres.

By *Statics*, Art 130, the upper body if solid would be in stable equilibrium if the height of its centre of gravity above A were such that

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{2r}, \text{ i.e. } > \frac{3}{2r},$$

i.e. if $$h < \frac{2r}{3}.$$

Now when the upper hemisphere is turned slightly the water thrust on it still passes through the centre O'. Hence, as far as the question of stability is concerned, we may replace the water by an equal weight at O'.

Thus, if W' be the weight of the water and W that of the hemisphere, we have, since the C.G. of the latter bisects AO',

$$\frac{W' \times r + W \times \frac{r}{2}}{W' + W} < \frac{2r}{3}, \text{ i.e. } 3W' + \frac{3W}{2} < 2W' + 2W.$$

Hence $$W' < \frac{W}{2}.$$

EXAMPLES. XVII. (Page 95.)

1. wt. of water $= 187 \cdot 63 - 7 \cdot 95 = 179 \cdot 68$ grs.

wt. of liquid $= 142 \cdot 71 - 7 \cdot 95 = 134 \cdot 76$ grs.

$$\therefore \text{ required sp. gr.} = \frac{134 \cdot 76}{179 \cdot 68} = \frac{3}{4} = \cdot 75.$$

2. $$\text{sp. gr.} = \frac{\text{wt. of alcohol}}{\text{wt. of water}} = \frac{773}{983} = \cdot 7864, \text{ nearly.}$$

3. wt. of the water displaced by the iron $= (44 + 10 - 52\cdot 7)$ grms.
$= 1\cdot 3$.

$$\therefore \text{ sp. gr. of the iron} = \frac{10}{1\cdot 3} = 7\tfrac{9}{13} = 7\cdot 6923\ldots.$$

4. wt. of the water displaced by the solid
$= (38\cdot 4 + 22\cdot 3 - 49\cdot 8)$ grammes $= 10\cdot 9$ grammes.

$$\therefore \text{ sp. gr. of the solid} = \frac{22\cdot 3}{10\cdot 9} = 2\cdot 0458\ldots.$$

5. wt. of the water displaced
$= (212 + 50 - 254)$ grains $= 8$ grains.

$$\therefore \text{ sp. gr. of the metal} = \frac{50}{8} = 6\cdot 25.$$

6. Let w, w', w'' have the same meanings as in Art. 72 (1), and let σ_1 be the real sp. gr. of the liquid, and D that of the substance of which the "weights" are made. Then

$$w'\left(1 - \frac{a}{D}\right) = \text{wt. in air of the bottle} + \text{wt. of the water} \times (1 - a),$$

$$w''\left(1 - \frac{a}{D}\right) = \text{,, \quad ,, \quad ,, \quad ,,} + \text{wt. of the liquid} \times \left(1 - \frac{a}{\sigma_1}\right),$$

$$w\left(1 - \frac{a}{D}\right) = \text{,, \quad ,, \quad ,, \quad ,,} .$$

$$\therefore \frac{w'' - w}{w' - w} = \frac{\text{wt. of the liquid}}{\text{wt. of the water}} \times \frac{1 - \dfrac{a}{\sigma_1}}{1 - a} = \sigma_1 \times \frac{1 - \dfrac{a}{\sigma_1}}{1 - a}.$$

$$\therefore \sigma = \frac{\sigma_1 - a}{1 - a}.$$

$$\therefore \sigma_1 = a + \sigma(1 - a) = \sigma - a(\sigma - 1).$$

EXAMPLES. XVIII. (Pages 100, 101.)

1. wt. of water displaced by the body
$= (732 - 252)$ grammes $= 480$ grammes.

$$\therefore \text{ sp. gr.} = \frac{732}{480} = 1\cdot 525.$$

2. $\text{sp. gr.} = \dfrac{\text{wt.}}{\text{wt. of water displaced}} = \dfrac{2 \cdot 4}{2 \cdot 4 - 1 \cdot 6} = 3.$

3. $\text{sp. gr.} = \dfrac{\text{wt.}}{\text{wt. of water displaced}} = \dfrac{\text{wt.}}{\text{wt. of turp. displaced}}$

$$\times \frac{\text{wt. of turp. displaced}}{\text{wt. of water displaced}}$$

$$= \frac{3}{3 - 1 \cdot 86} \times \frac{\cdot 88}{1} = \frac{3}{1 \cdot 14} \times \cdot 88 = 2\tfrac{6}{19}.$$

4. wt. of naphtha displaced $= 432 \cdot 5 - 9 = 423 \cdot 5$ grms.;

$$\therefore \text{ sp. gr.} = \frac{432 \cdot 5}{423 \cdot 5} \times \text{sp. gr. of naphtha} = \frac{4325}{4235} \times \cdot 847 = \cdot 865.$$

5. wt. of wood and lead in water $-$ wt. of lead in water

$$= 20 - 30 = -10 \text{ grains.}$$

$\therefore$ 120 grains $-$ wt. of water displaced by wood $= -10$ grains.

$\therefore$ wt. water displaced by wood $= 130$ grains.

$$\therefore \text{ sp. gr.} = \frac{120}{130} = \frac{12}{13}.$$

6. wt. of solid in water $= 6 - 8 = -2$ lbs.

$\therefore$ wt. water displaced by solid $= 4 - (-2) = 6$ lbs.

$$\therefore \text{ sp. gr.} = \frac{4}{6} = \frac{2}{3}.$$

7. $200 = 200\left(1 - \dfrac{1}{\sigma}\right) + 300\left(1 - \dfrac{1}{5}\right).$

$$\therefore \frac{2}{\sigma} = 3 \times \frac{4}{5}. \qquad \therefore \sigma = \frac{5}{6}.$$

8. $22 = 47\left(1 - \dfrac{1}{\sigma}\right)$ and $25 \cdot 8 = 47\left(1 - \dfrac{\sigma'}{\sigma}\right)$, where σ, σ' are the sp. grs. of glass and alcohol.

$$\therefore \frac{1}{\sigma} = \frac{25}{47} \text{ and } \frac{\sigma'}{\sigma} = \frac{21 \cdot 2}{47}.$$

$$\therefore \sigma' = \frac{21 \cdot 2}{25} = \cdot 848.$$

9.
$$1 = 1{\cdot}09\left[1 - \frac{\sigma}{11{\cdot}4}\right].$$

$$\therefore \quad \frac{\sigma \times 1{\cdot}09}{11{\cdot}4} = {\cdot}09.$$

$$\therefore \quad \sigma = \frac{9 \times 11{\cdot}4}{109} = {\cdot}9413, \text{ nearly.}$$

10. wt. of water displaced by the glass $= 665{\cdot}8 - 465{\cdot}8$
$= 200$ grammes.

wt. of sulphuric acid displaced by it $= 665{\cdot}8 - 297{\cdot}6$
$= 368{\cdot}2$ grammes.

$$\therefore \text{ sp. gr. required} = \frac{368{\cdot}2}{200} = 1{\cdot}841.$$

11. wt. of water displaced by sugar and wax
$= 40 + 5{\cdot}76 - 14{\cdot}76 = 31$ grammes.

$$\text{wt. of water displaced by wax} = \frac{5{\cdot}76}{{\cdot}96} = 6 \text{ grammes.}$$

$\therefore$ wt. of water displaced by sugar $= 25$ grammes.

$$\therefore \text{ sp. gr.} = \frac{40}{25} = 1{\cdot}6.$$

12. wt. of water displaced by copper and wax
$= 72 + 18 - 62 = 28$ grammes.

$$\text{wt. of water displaced by wax} = \frac{18}{{\cdot}9} = 20 \text{ grammes,}$$

wt. of water displaced by copper $= 8$ grammes.

$$\therefore \text{ sp. gr.} = \frac{72}{8} = 9.$$

13. If V cub. cms. be the volume of the marble and σ the required sp. gr., then

$$92 = V(2{\cdot}84 - 1) \text{ and } 98{\cdot}5 = V(2{\cdot}84 - \sigma).$$

$$\therefore \quad \frac{92}{98{\cdot}5} = \frac{1{\cdot}84}{2{\cdot}84 - \sigma}, \quad i.e. \quad \frac{1}{98{\cdot}5} = \frac{2}{284 - 100\sigma}.$$

$$\therefore \quad 100\sigma = 284 - 197 = 87.$$

$$\therefore \quad \sigma = {\cdot}87. \quad \text{Also } V = \frac{92}{1{\cdot}84} = 50.$$

14. Let V cub. cms. and σ be its volume and sp. gr. Then

$$18 = V(\sigma - \cdot 8) \text{ and } 12 = V(\sigma - 1\cdot 2).$$

$$\therefore \frac{3}{2} = \frac{\sigma - \cdot 8}{\sigma - 1\cdot 2}, \text{ i.e. } \sigma = 3\cdot 6 - 1\cdot 6 = 2.$$

Also
$$V = \frac{18}{\sigma - \cdot 8} = \frac{18}{1\cdot 2} = 15.$$

$$\therefore \text{wt.} = V\sigma = 30 \text{ grammes.}$$

15. Let W, W' be the true wts. of the sinker and substances and σ, σ' their sp. grs. Then

$$\left.\begin{aligned} W\left(1 - \frac{1}{\sigma}\right) &= 5W', \\ W\left(1 - \frac{1}{\sigma}\right) + W'\left(1 - \frac{1}{\sigma'}\right) &= 4W' \end{aligned}\right\}.$$

$\therefore$ by subtraction
$$W'\left(1 - \frac{1}{\sigma}\right) = -W'.$$

$$\therefore \sigma = \frac{1}{2}.$$

16. s and s' being the sp. grs. of the body and the liquid, then

$$W = 4W\left(1 - \frac{1}{s}\right) \text{ and } W\left(1 - \frac{1}{s}\right) = \frac{4}{3}W\left(1 - \frac{s'}{s}\right),$$

$$\therefore 1 = 4 - \frac{4}{s} \text{ and } 1 - \frac{1}{s} = \frac{4}{3} - \frac{4}{3}\frac{s'}{s}.$$

$$\therefore s = \frac{4}{3} \text{ and } s' = \frac{13}{12}.$$

17. wt. water displaced $= \dfrac{7\frac{1}{2}}{19\cdot 2}$ lbs.

$$\therefore \text{wt. in water} = 7\tfrac{1}{2} - \frac{7\frac{1}{2}}{19\cdot 2} = \frac{182}{192} \times 7\tfrac{1}{2} = 7\tfrac{7}{64} \text{ lbs.}$$

If it contained x lbs. of gold and y lbs. of silver, then

$$x + y = 7\tfrac{1}{2},$$

and
$$\frac{x}{19\cdot 2} + \frac{y}{10\cdot 5} = 7\tfrac{1}{2} - 7\tfrac{7}{64} = \frac{8}{17}.$$

$\therefore$ solving, we get
$$x = \frac{96}{17} \text{ and } y = \frac{63}{34}.$$

18. W and w being as in Page 96, we have, if σ_1 be the true spec. grav. and D that of the "weights,"

$$w\left(1-\frac{a}{D}\right)=\text{wt. of body}-\text{wt. of displaced water},$$

$$W\left(1-\frac{a}{D}\right)=\text{wt. of the body}\times\left(1-\frac{a}{\sigma_1}\right).$$

$$\therefore\ \frac{w}{W}=\frac{1}{1-\frac{a}{\sigma_1}}-\frac{1}{1-\frac{a}{\sigma_1}}\times\frac{\text{wt. of displaced water}}{\text{wt. of the body}}\ \ldots\ldots\ (1).$$

But
$$\frac{W}{W-w}=\sigma,\quad i.e.\ \frac{1}{\sigma}=1-\frac{w}{W}.$$

$\therefore$ (1) gives
$$1-\frac{1}{\sigma}=\frac{1}{1-\frac{a}{\sigma_1}}\left(1-\frac{1}{\sigma_1}\right)=\frac{\sigma_1-1}{\sigma_1-a}.$$

$$\therefore\ \sigma_1\left[1-\left(1-\frac{1}{\sigma}\right)\right]=1-a\left(1-\frac{1}{\sigma}\right).$$

$$\therefore\ \sigma_1=\sigma-a(\sigma-1).$$

EXAMPLES. XIX. (Pages 108, 109.)

1. The volumes V_1, V_2, V_3 cubic ft. are given by

$$\frac{2}{16}=V_1\times w=V_2\times\frac{11}{10}w=V_3\times\frac{12}{10}w.$$

$$\therefore\ \frac{2}{16}\times\frac{2}{125}=V_1=V_2\times\frac{11}{10}=V_3\times\frac{12}{10}.$$

$$\therefore\ V_1=\frac{2}{1000}\text{ cub. ft.}=3\cdot456\text{ cub. ins.}$$

$$V_2=\frac{20}{11000}\text{ cub. ft.}=\frac{3456}{1100}=3\cdot1\dot{4}\dot{1}\dot{8}\text{ cub. ins.}$$

$$V_3=\frac{1}{600}\text{ cub. ft.}=\frac{1728}{600}=2\cdot88\text{ cub. ins.}$$

2.
$$W=\frac{9}{10}V.1.w=\frac{90}{103}.V.\sigma.w.$$

$$\therefore\ \sigma=\frac{9}{10}\times\frac{103}{90}=1\cdot03.$$

3. $$W=(V-4A)\times 1\cdot 2w=(V-8A)\times 1\cdot 4w$$
$$=(V-xA)\times 1\cdot 3w.$$
$$\therefore\ 2V=112A-48A=64A,$$
and $$V=13xA-48A=(13x-48)A.$$
$$\therefore\ 2(13x-48)=64.$$
$$\therefore\ 13x=48+32=80,\quad i.e.\ x=6\tfrac{2}{13}\text{ ins.}$$

4. $$W=V\times 1\cdot 6w=(V+A)\times 1\cdot 3w=(V+2A)\times xw.$$
$$\therefore\ 3V=13Aw,$$
and $$\left.\begin{array}{r}\\ (16-10x)V=20Axw\end{array}\right\}.$$
$$\therefore\ 60Axw=13A(16-10x).$$
$$\therefore\ 190x=208.$$
$$\therefore\ x=1\tfrac{18}{190}=1\cdot 0947.$$

5. $$9=(12-x)\times\cdot 85.$$
$$\therefore\ 85x=1020-900=120.$$
$$\therefore\ x=\frac{24}{17}=1\tfrac{7}{17}\text{ cub. cms.}$$

6. Let W be the original wt. and xW the part lost. Then
$$W=V\times 1\cdot 002w,$$
and $$W-xW=V\times w.$$
$$\therefore\ 1-x=\frac{1}{1\cdot 002}.$$
$$\therefore\ x=1-\frac{1}{1\cdot 002}=\frac{\cdot 002}{1\cdot 002}=\frac{1}{501}.$$

7. $$W=(V-2A)\times\cdot 8w=(V-3A)\times\cdot 85w=(V-4A)\times xw.$$
$$\therefore\ 5V=255A-160A=95A,$$
$$(10x-8)V=40Ax-16A=(40x-16)A.$$
$$\therefore\ \frac{10x-8}{5}=\frac{40x-16}{95}.$$
$$\therefore\ x=\frac{272}{300}=\cdot 90\dot{6}.$$

8. $$10 \text{ ozs.} = V . s_1,$$
and $$13 \text{ ozs.} = V . s_2.$$
$$\therefore \; s_1 : s_2 :: 10 : 13.$$

9. $$4\tfrac{3}{4} + 2 = V . s_1,$$
and $$4\tfrac{3}{4} + 2\tfrac{3}{8} = V . s_2.$$
$$\therefore \; s_1 : s_2 :: 6\tfrac{3}{4} : 7\tfrac{1}{8}$$
$$:: 54 : 57$$
$$:: 18 : 19.$$

10. $$3\tfrac{3}{4} + 1\tfrac{3}{4} = V . 1 . w,$$
and $$3\tfrac{3}{4} + x = V \times 2{\cdot}2 \times w.$$
$$\therefore \; 5\tfrac{1}{2} \times 2{\cdot}2 = 3\tfrac{3}{4} + x.$$
$$\therefore \; x = \frac{11}{2} \times \frac{11}{5} - \frac{15}{4} = \frac{121}{10} - \frac{15}{4}$$
$$= 12\tfrac{1}{10} - 3\tfrac{3}{4} = 8\tfrac{7}{20}.$$

11. Let W = wt. of hydrometer, W_1 = wt. of solid, and s = required sp. gr.

Then
$$W + W_1 + 12 = W + W_1\left(1 - \frac{1}{s}\right) + 16 = W + 22$$
$$= \text{wt. water displaced.}$$
$$\therefore \; \left.\begin{aligned} W_1 &= 10, \\ \text{and} \quad \frac{W_1}{s} &= 4 \end{aligned}\right\} \quad \therefore \; s = \frac{10}{4} = 2{\cdot}5.$$

12. Let W = wt. of hydrometer, W_1 = wt. of substance, and s = required sp. gr.

Then
$$W + 1250 = W + W_1 + 530 = W + W_1\left(1 - \frac{1}{s}\right) + 620$$
$$= \text{wt. of the displaced water.}$$
$$\therefore \; W_1 = 720 \text{ and } \frac{W_1}{s} = 90.$$
$$\therefore \; s = \frac{720}{90} = 8.$$

13. Here

$$2 = x\left(1 - \frac{1}{8}\right).$$

$$\therefore\ x = \frac{16}{7} = 2\tfrac{2}{7}\text{ ozs.}$$

15. Let W be the weight of the hydrometer; when V, V' are the volumes of the immersed and unimmersed portions, let σ_1 be the marked sp. gr. so that

$$W = V.\sigma_1.$$

When it is used in air, let it sink to the same level in a liquid of sp. gr. σ_2.

$$\therefore\ W = V\sigma_2 + V'\sigma.$$

$$\therefore\ V\sigma_1 = V\sigma_2 + V'\sigma.$$

$$\therefore\ \sigma_1 = \sigma_2 + \frac{V'}{V}\sigma.$$

$$\therefore\ \sigma' = \frac{V'}{V}\sigma,\quad i.e.\ \frac{\sigma'}{\sigma} = \frac{V'}{V}.$$

16. Let W and V be the original wt. and volume, and W', V' the wt. and vol. of the part chipped off.

Let xA be the volume out of the liquid down to the mark which properly corresponded to a density α'. Then

$$W = (V - Ax)\,\alpha', \text{ and } W - W' = (V - V' - Ax)\,\alpha.$$

$$\therefore\ \frac{W}{\alpha'} - \frac{W - W'}{\alpha} = V'.$$

So

$$\frac{W}{\beta'} - \frac{W - W'}{\beta} = V',$$

and

$$\frac{W}{\gamma'} - \frac{W - W'}{\gamma} = V'.$$

Hence, by subtraction,

$$W\left[\frac{1}{\alpha'} - \frac{1}{\alpha} - \frac{1}{\beta'} + \frac{1}{\beta}\right] + W'\left[\frac{1}{\alpha} - \frac{1}{\beta}\right] = 0,$$

and

$$W\left[\frac{1}{\alpha'} - \frac{1}{\alpha} - \frac{1}{\gamma'} + \frac{1}{\gamma}\right] + W'\left[\frac{1}{\alpha} - \frac{1}{\gamma}\right] = 0.$$

$$\therefore\ \left(\frac{1}{\alpha'} - \frac{1}{\alpha} - \frac{1}{\beta'} + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\gamma}\right)$$

$$= \left(\frac{1}{\alpha'} - \frac{1}{\alpha} - \frac{1}{\gamma'} + \frac{1}{\gamma}\right)\left(\frac{1}{\alpha} - \frac{1}{\beta}\right),$$

$$\therefore \frac{1}{\alpha'\beta} - \frac{1}{\alpha\beta'} = \frac{1}{\gamma}\left[\frac{1}{\alpha'} - \frac{1}{\beta'}\right] - \frac{1}{\gamma'}\left[\frac{1}{\alpha} - \frac{1}{\beta}\right],$$

and hence, on reduction,

$$\gamma' = \frac{\gamma\beta'\alpha'(\alpha-\beta)}{\gamma(\alpha\beta' - \alpha'\beta) + \alpha\beta(\alpha' - \beta')},$$

and
$$\gamma = \frac{\gamma'\alpha\beta(\alpha'-\beta')}{\gamma'(\alpha'\beta - \alpha\beta') + \alpha'\beta'(\alpha-\beta)}.$$

EXAMPLES. XX. (Pages 111, 112.)

1. If the mercury rises one inch in one leg it must fall one inch in the other, so that the level of the mercury in the latter leg is two inches below that in the former. Hence, if x is the required depth of water,

$$2 \times 13\cdot6 \times w = x \times w.$$
$$\therefore\ x = 27\cdot2.$$

2. Let the water have risen x inches in one leg when the other leg is filled with oil. Then as in the last example

$$2x \times w = (x+4) \times \frac{2}{3} w.$$

$\therefore\ x = 2$, and amount of oil poured in $= x + 4 = 6$ inches.

3. At the bottom of the vertical tube containing the oil; for then there are 4 inches of water in one tube and 5 in the other, and the pressures due to these depths are the same.

4. The junction of the mercury and liquid in one leg is one inch below the level of the mercury in the other, and 8 inches below the level of the liquid.

$$\therefore\ 8 \times \sigma = 1 \times 13\cdot6.$$
$$\therefore\ \sigma = 1\cdot7.$$

5. If the mercury rises 1 inch in one tube the other surface of the mercury falls $\frac{1}{10}$ inch and thus the difference between the levels of the two ends of the mercury is $\frac{11}{10}$ inch.

Hence, if x be the ht. of the water in inches,

$$x \times 1 = \frac{11}{10} \times 13\cdot596,$$

$\therefore$ amount of water poured in $= x$ cub. ins.

$$= \frac{11}{10} \times 13\cdot596 = 14\cdot9556 \text{ cub. ins.}$$

6. If the surface of the mercury in the longer arm is lowered by x cms., it is raised by $2x$ in the other, so that the difference between the levels is $3x$. Hence, since the 52 cub. cms. will occupy a length 26 cms. in the longer arm,

$$26 = 3x \times 13{\cdot}65.$$

$$\therefore\ x = \frac{26}{3 \times 13{\cdot}65} = \frac{40}{63} \text{ cms.}$$

7. Before the second liquid is poured in let there be $y + x$ of the first liquid in the one leg and y in the other, so that

$$b\sigma = x\rho \ \ldots\ldots\ldots\ldots (1),$$

and the length of each leg is thus $y + x + c$. When the second liquid has been poured in to a depth z, let there be y'' of the first liquid in its leg, and y' in the other leg, the unoccupied part in it being now c'.

$$\therefore\ y' + c' = y'' + b + z = y + x + c \ \ldots\ldots\ldots\ldots (2).$$

Also $$y' + y'' = \text{total amount of 1st liquid}$$
$$= 2y + x \ \ldots\ldots\ldots\ldots (3).$$

(2) and (3) give $$x = b + c' + z - 2c \ \ldots\ldots\ldots\ldots (4).$$

Also, by equating pressures, we have

$$\sigma b + \tau z = \rho\,(y' - y'') = \rho\,(b + z - c') \ \ldots\ldots\ldots\ldots (5).$$

(4) and (5) give

$$\rho\,[x + 2c - 2c'] = \sigma b + \tau\,[x - b - c' + 2c],$$

i.e. by (1)

$$\rho\,(2c - 2c') = \tau\left[\frac{b\sigma}{\rho} - b - c' + 2c\right].$$

$$\therefore\ c'\,[2\rho - \tau] = 2c\,(\rho - \tau) + b\tau\frac{\rho - \sigma}{\rho}.$$

$$\therefore\ \text{etc.}$$

EXAMPLES. XXI. (Page 126.)

1. The height $= \dfrac{13{\cdot}596 \times 77{\cdot}4}{{\cdot}9}$ cms.

$$= 1169{\cdot}256 \text{ cms.}$$

2. Thrust $=$ wt. of $\pi \,.\, 7^2 \times (5000 + 1033)$ cub. cms. of water

$$= \frac{22}{7} \times 7^2 \times 6033 \text{ grammes wt.}$$

$$= 154 \times 6033 \text{ grammes wt.}$$

$$= 929082 \text{ grammes wt.}$$

3. Sp. gr. of glycerine $=\frac{13\cdot6\times2\frac{1}{2}}{26}=1\frac{4}{13}$.

Wt. of bullet + actual ht. × section of tube × σw

$$= \text{true ht.} \times \text{section of tube} \times \sigma w,$$

where $\sigma=$ sp. gr. of mercury, so that, if

$$\text{wt. of bullet} = \text{section of tube} \times x \times \sigma w,$$

then actual ht. = true ht. $-x$.

4. The cistern level falls through a distance

$$=\frac{2\cdot5\times1^2}{(4\cdot5)^2}=\frac{10}{81}\text{ cm.}$$

$$\therefore \text{ real alteration in the ht.} = 2\cdot5+\frac{10}{81}=2\cdot623\ldots\text{ cms.}$$

5. The surface in the cistern falls a distance

$$=\frac{\left(\frac{1}{6}\right)^2\times1}{\left(\frac{3}{2}\right)^2}=\frac{1}{81}\text{ inch.}$$

$$\therefore \text{ real alteration} = 1\tfrac{1}{81}\text{ inch.}$$

EXAMPLES. XXII. (Pages 135—138.)

1. Required sp. gr. $=\frac{760}{700}\times\cdot00119$. For the pressure of the air supports columns of mercury in the two cases of lengths 760 and 700 mm. respectively.

$$\therefore \text{ sp. gr.} = 76\times\cdot000017=\cdot001292.$$

2. $$\text{New wt.} = 310\times\frac{30\cdot23}{29\cdot45}=2\times\frac{30\cdot23}{\cdot19}$$

$$=\frac{6046}{19}=318\tfrac{4}{19}\text{ grs. wt.}$$

$$\therefore \text{ etc.}$$

3. If x be the depth in feet then, by Boyle's Law,

$$3\times(33+10)=2\times(33+x).$$

$$\therefore x=31\tfrac{1}{2}\text{ ft.}$$

4. If x be the required depth of the surface of the water inside the tumbler the pressure of the air there is that due to a depth $x+h$ of water and thus Boyle's Law gives

$$1 \times h = \frac{1}{3} \times (x+h).$$

$$\therefore\ x = 2h.$$

In the case of the conical wine-glass, since volumes of similar cones are as the cubes of their heights, the new volume of the air

$$= \frac{1}{2^3} \times \text{original volume}.$$

Hence $$h \times 1 = (x+h) \times \frac{1}{2^3}.$$

$$\therefore\ x = 7h.$$

5. Boyle's Law gives

$$1 \times h = \frac{1}{2} \times (h + 32{\cdot}75).$$

$$\therefore\ h = 32{\cdot}75 \text{ ft.}$$

6. Originally the level of the water outside and inside was the same. Let the tube be raised x cms. Then the level of the water inside is $x+25-50$, *i.e.* $x-25$ cms. above that outside. Hence the pressure of the contained air is that due to $76-(x-25)$ cms. of water, *i.e.* to $101-x$.

Hence Boyle's Law gives

$$25 \times 76 = 50 \times (101 - x).$$

$$\therefore\ x = 63.$$

7. If there were no vent-peg to the barrel the air inside the barrel would be cut off from communication with the external air. As the beer was drawn out this confined air would expand in volume and decrease in pressure, until finally we might arrive at a position at which the pressure of the beer at the tap due to the depth of the beer inside the barrel and the contained air was less than the pressure of the atmosphere. In this case the pressure of the atmosphere would keep the beer from flowing out at the tap.

Similarly for the teapot if the lid fitted quite tightly and had no hole.

8. What at atmospheric pressure would occupy 5 feet is compressed till it occupies $1\frac{1}{3}$ feet. Hence the pressure on one side

$$= \frac{5}{1\frac{1}{3}} \times 15 \text{ lbs. per sq. inch}$$

$$= 56\tfrac{1}{4} \text{ lbs. per sq. inch.}$$

On the other side what would occupy 1 foot now occupies $\frac{2}{3}$ ft.

Hence its pressure $=\frac{3}{2}\times 15=22\frac{1}{2}$ lbs. per sq. inch.

When there is equilibrium let the piston be distant x inches from the centre. Then

$$\frac{5\times 15}{1+\frac{x}{12}}=\frac{1\times 15}{1-\frac{x}{12}}.$$

$$\therefore\ x=8.$$

9. Since the pressure of the atmosphere decreases the coal-gas expands till it again displaces its own weight of air and floats.

If it had been quite full originally it could no longer expand and thus as it no longer displaces as much as its own wt. of air, it would sink.

10. Let x be the wt. of counterpoise. Then

$$60-x=\pi\,.\,(1\tfrac{1}{4})^2\times p,$$

where
$$p=\frac{1}{12}\times w.$$

$$\therefore\ x=60-\pi\,.\,\frac{25}{16}\,.\,\frac{62\frac{1}{2}}{12}=60-25{\cdot}6 \text{ nearly}$$
$$=34{\cdot}4 \text{ nearly}.$$

11. When lowered through the 11 ft. the air has a volume

$$=\frac{33}{33+11}=\frac{3}{4} \text{ of its original.}$$

The weight of the water displaced then

$$=\frac{3}{4}\times w=15 \text{ ozs.}$$

Also since wt. of bottle + 5 ozs. = wt. of water originally displaced = 20 ozs.

$$\therefore\ \text{wt. of bottle}=15 \text{ ozs.}$$

Hence in the lower position the wt. of the bottle is just balanced by the wt. of the displaced water. It will then just float.

If lowered the air becomes decreased in volume, and so the upward thrust of the displaced air becomes lessened and the bottle sinks.

Similarly if it be raised the upward thrust is increased and the bottle rises.

12. The air has its volume diminished from a length a of the cylinder to a length $a-k$, and its pressure therefore increased from hw to $h \cdot \frac{a}{a-k} \cdot w$.

Hence
$$(a+k)\,w+\frac{ha}{a-k}\,w=2\,[a+h]\,w.$$
$$\therefore\ a^2-k^2+ha=2a^2-2ak-2hk+2ha.$$
$$\therefore\ k^2-2k\,(h+a)+ha+a^2=0.$$
$$\therefore\ k=h+a\pm\sqrt{h^2+ah}.$$

The upper sign is impossible; for it would give a value to k greater than a, which is impossible.

13. Let σ be the sp. gr. of the iron and h the height of the water-barometer.

The new pressure of the contained air is $\frac{7h}{6}\,w$.

$$\therefore\ \frac{2}{3}\times\sigma w=\left(\frac{7h}{6}\,w-hw\right)=\frac{hw}{6}\ \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The final pressure of the air

$$=\frac{7hw}{7-1\frac{38}{41}}=\frac{287hw}{210}=\frac{41hw}{30}.$$
$$\therefore\ \frac{2}{3}\,\sigma w+6w=\left(\frac{41hw}{30}-hw\right)=\frac{11hw}{30}\ \ldots\ldots\ldots\ldots(2).$$
$$\therefore\ h=30 \text{ and } \sigma=\frac{h}{4}=7\tfrac{1}{2}.$$

14. The pressure of the air inside is that at a depth in water
$$=k-(h-x)=x+k-h.$$
$$\therefore\ x\,(x+k-h+H)=h\,.\,H, \text{ by Boyle's Law.}$$
$$\therefore\ x^2+x\,(H+k-h)=H\,.\,h.$$

15. Let the mercury have risen to a height x ins. Then the pressure of the air inside, being equal to that of the mercury which it touches, is equal to that of water at a depth $4\frac{7}{32}-x$.

Hence, by Boyle's Law,
$$(4\tfrac{7}{32}-x)\,(4\tfrac{7}{32}-x+30)=4\tfrac{7}{32}\times 30.$$
$$\therefore\ x=\frac{15}{32}, \text{ the other value, } \frac{1215}{32}, \text{ being inadmissible.}$$

16. Let x be the required height, so that the new volume of the air is $\left(\frac{x}{4}\right)^3 \times$ original volume.

$$\therefore \left(\frac{x}{4}\right)^3 [34+34] = 1 \times 34.$$

$$\therefore x = \frac{4}{\sqrt[3]{2}} = \sqrt[3]{32}.$$

17. Let V be the volume of the cone, x its height, and nx the height of the air inside the cone in the first case, so that

$$n^3V \times (nx+h) = V \times h \quad \text{........................(1)},$$

and

$$W = n^3V \times w \quad \text{....................(2)}.$$

Also

$$W + mW = V \times w \quad \text{........................(3)}.$$

(2) and (3) give

$$n^3 = \frac{1}{1+m},$$

and then (1) gives

$$nx + h = \frac{h}{n^3} = h(1+m).$$

$$\therefore x = \frac{mh}{n} = mh \,.\, \sqrt[3]{m+1}.$$

18. When the depth of the mouth is 13 feet, let the air inside the cylinder occupy a length x, so that

$$x[33+13-8+x] = 8 \times 33,$$

and hence $x = 6$.

Let A be the internal cross section, and therefore $\frac{3}{4}A$ the cross section of the iron.

The volume of the water displaced then

$$= \frac{3A}{4} \times 8 + A \times 6 = 12A,$$

and the upward thrust of the water $= 12Aw$.

The weight of the cylinder

$$= \frac{3A}{4} \times 8 \times 2w = 12Aw.$$

The cylinder is now just in equilibrium. If it be lowered further, the air is still further compressed, the upward thrust is lessened, and the cylinder sinks by itself.

19. If Π be the atmospheric pressure, then when the piston has sunk 2 feet the pressure of the air under it $=\frac{5}{3}\Pi$; hence, if A be the area of the piston,

$$30=A\left[\frac{5\Pi}{3}-\Pi\right]=\frac{2A\Pi}{3} \quad \text{.....................(1).}$$

When the piston has sunk another 2 feet the pressure of the contained air

$$=\frac{5}{1}\Pi=5\Pi.$$

Hence, if x be the required force in lbs. wt.

$$x+30=A\,[5\Pi-\Pi]=4A\Pi=6\times 30=180.$$

$$\therefore\ x=150.$$

20. At atmospheric pressure let the air in the cylinder occupy a length x of it. Then its pressure now

$$=\frac{\Pi x}{1}=\Pi x.$$

$$\therefore\ \pi\,.\left(\frac{1}{2}\right)^2\times[\Pi x-\Pi]=62{\cdot}5\pi+\pi\,.\left(\frac{1}{2}\right)^2.\,3w\text{... ....(1).}$$

When the sphere is immersed let the piston descend y feet, so that the new pressure of the air

$$=\frac{\Pi x}{1-y}.$$

The effect of the immersion is to increase the thrust on the piston by the weight of a volume of water equal to that of the sphere, *i.e.* by

$$\frac{4}{3}\pi\left(\frac{3}{8}\right)^3.\,w.$$

$$\therefore\ \pi\,.\left(\frac{1}{2}\right)^2.\left[\frac{\Pi x}{1-y}-\Pi\right]=62{\cdot}5\pi+\pi\,.\left(\frac{1}{2}\right)^2.\,3w+\frac{4}{3}\pi\,.\left(\frac{3}{8}\right)^3 w\text{...(2).}$$

Hence if $\Pi=wh$ and $w=62{\cdot}5$, these equations give

$$h\,(x-1)=7,$$

and $$h\left[\frac{x}{1-y}-1\right]=7\tfrac{9}{32}.$$

$$\therefore\ 7\tfrac{9}{32}\,[1-y]-7=hy,$$

i.e. $$y=\frac{\frac{9}{32}}{h+7\frac{9}{32}}=\frac{9}{32h+233}.$$

21. Let x be the distance from the centre at which the piston rests, so that the pressures of the air above and below it are

$$\frac{\Pi a}{a+x} \text{ and } \frac{\Pi a}{a-x}.$$

$$\therefore\ A\left[\frac{\Pi a}{a-x}-\frac{\Pi a}{a+x}\right]=W\sin\alpha.$$

$$\therefore\ \lambda a\,.\,\frac{2x}{a^2-x^2}=\sin\alpha.$$

$$\therefore\ x^2+2\lambda a\operatorname{cosec}\alpha\,.\,x=a^2.$$

$$\therefore\ x=-\lambda a\operatorname{cosec}\alpha+\sqrt{a^2+\lambda^2a^2\operatorname{cosec}^2\alpha}.$$

23. Let the water be on the point of running over when a quantity has been poured in that would fill a length x of the cylinder. The pressure of the confined air is then

$$\Pi\frac{h}{h-x},\quad i.e.\quad wH\,.\,\frac{h}{h-x}.$$

For the equilibrium of the piston we then have

$$A\,.\,wH\frac{h}{h-x}=A\,(wx+wH).$$

$$\therefore\ hH=(h-x)(x+H),$$

i.e.

$$x=h-H.$$

If H be $>h$, then x is negative, showing that no water can be poured in without its overflowing. In this case if the piston were displaced downwards through a distance z, and then water poured in, then if $h=H-y$ the upward thrust on the piston

$$=A\frac{Hh}{h-z}=A\frac{h(h+y)}{h-z}=A\left[h+\frac{h}{h-z}(z+y)\right],$$

and downward thrust

$$=A[z+H]=A[h+(z+y)].$$

The first is therefore the greater, so that the piston would be lifted up and the water above it made to overflow.

EXAMPLES. XXIII. (Pages 141, 142.)

1. (i)

$$\frac{V\,.\,76}{1}=\frac{100\times 80}{1+\alpha\,.\,30}.$$

$$\therefore\ V=\frac{8000}{76}\times\frac{1}{1+\frac{30}{273}}=\frac{8000\times 273}{76\times 303}=\frac{182000}{1919}=94{\cdot}84\ldots$$

(ii)

$$100^\circ\text{ F.}=\frac{5}{9}(100-32)^\circ\text{ C.}=\frac{340^\circ}{9}\text{ C.}$$

Also 3 atmospheres $=3\times 76$ cms. of mercury.

$$\therefore\ V\times 76=\frac{3\times 3\times 76}{1+\frac{340}{9}\times\frac{1}{273}}.$$

$$\therefore\ V=\frac{9\times 9\times 273}{2797}=\frac{22113}{2797}=7\cdot 90\ldots \text{ cub. ft.}$$

2. $$\frac{x\times 51}{1+a\,.\,16}=\frac{9\times 57}{1+a\,.\,69}.$$

$$\therefore\ x=\frac{171}{17}\cdot\frac{1+\frac{16}{273}}{1+\frac{69}{273}}=\frac{171}{17}\times\frac{289}{342}=8\tfrac{1}{2}\text{ cub. ins.}$$

3. $$\frac{x\times 54}{1+\frac{78}{273}}=\frac{15\times 32}{1+\frac{39}{273}}.$$

$$\therefore\ x=\frac{15\times 32}{54}\times\frac{351}{312}=10\text{ cub. ins.}$$

4. Let V be the volume at the top of the mountain of the air which occupies a cubic metre at the bottom. Then

$$\frac{400\times V}{1+\frac{13}{273}}=\frac{750\times 1}{1+\frac{7}{273}}.$$

$$\therefore\ V=\frac{750}{400}\times\frac{286}{280}=\frac{429}{224}.$$

Hence $\frac{429}{224}$ c.c. weighs as much at the top as 1 c.c. at the bottom.

Therefore the weights of equal volumes at the top and bottom are as 224 : 429.

5. The original lengths occupied are $\frac{2l}{3}$ and $\frac{l}{3}$.

When the temperature is raised let the piston move x. Then, if P be the original pressure, the final pressures are

$$P(1+at)\frac{\frac{2l}{3}}{\frac{2l}{3}+x},\ \text{and}\ P\frac{\frac{l}{3}}{\frac{l}{3}-x};$$

equating these two pressures, we have

$$x=\frac{2lat}{9+6at}.$$

6. If the radius be doubled, the volume becomes multiplied by 2^3.

$$\therefore \frac{p \cdot 1}{1} = \frac{p' \cdot 2^3}{1 + a \cdot 455}.$$

$$\therefore p' = \frac{p}{8}\left[1 + \frac{455}{273}\right] = \frac{p}{8}\left[1 + \frac{5}{3}\right] = \frac{p}{3}.$$

7. Here p = pressure per sq. cm.

$$= 13 \cdot 596 \times 76 \times 981.$$

$$V = \frac{1}{\cdot 001} \text{ cub. cm.} = 1000 \text{ cub. cms.}$$

and $$T = 80 + \frac{1}{\cdot 00366} = \frac{129280}{366}.$$

$$\therefore \frac{pV}{T} = \frac{13 \cdot 596 \times 76 \times 981 \times 1000}{\frac{129280}{366}}$$

$$= 13596 \times 76 \times 981 \times \frac{366}{129280}$$

$$= 2870000 \text{ nearly.}$$

8. Let the length of the cylinder be $2a$, so that the distances from the ends of the cylinder of the piston are $\sqrt{2}a$ and $2a - \sqrt{2}a$. Hence the pressures of the air are $\frac{\Pi}{\sqrt{2}}$ and $\frac{\Pi}{2-\sqrt{2}}$. Hence, if A be the area of the piston and W its weight,

$$W = A\left[\frac{\Pi}{2-\sqrt{2}} - \frac{\Pi}{\sqrt{2}}\right] = \Pi \cdot A \text{ ....(i).}$$

The temperature being raised from T to t_1 and t_2 respectively the new pressures are

$$\frac{\Pi \cdot a}{T} \div \frac{a}{t_1} = \Pi \frac{t_1}{T},$$

and $$\Pi \frac{t_2}{T}.$$

$$\therefore \Pi\left[\frac{t_1}{T} - \frac{t_2}{T}\right] \times A = W = \Pi \cdot A.$$

$$\therefore t_1 - t_2 = T.$$

Page 147.

Ex. 1. Required height $= \dfrac{k}{g \log_{10} e}$ [log 76 − log 75]

$$= \frac{779741000}{981 \times \cdot 43429} \times \cdot 00575 \text{ cms.} = \text{about } 106 \text{ metres.}$$

Ex. 2. $\dfrac{\text{Required height}}{4500} = \dfrac{\log 30 - \log 20}{\log 30 - \log 25} = \dfrac{\log 3 - \log 2}{\log 3 + 2 \log \overline{2} - \overline{1}}$

$$= \frac{\cdot 1760913}{\cdot 0791813}. \quad \therefore \text{ etc.}$$

EXAMPLES. XXIV. (Pages 149—151.)

1. The iron merely depresses the column through a distance x such that the volume of a length x of the tube of mercury equals the weight of the iron. As mercury has nearly twice the sp. gr. of iron this depression is very small. On the other hand the air expands (and so forces the mercury down) until its pressure differs from that of the external air by a quantity which is measured by the then height of the barometer.

2. Let the length of the original vacuum be x inches, so that the length of the barometer tube is $(x+30)$ ins., and hence the length occupied by the enclosed air finally $= x + 30 - 26 = x + 4$. This air is at a pressure $30 - 26 = 4$ ins.

Therefore Boyle's Law gives

$$(x+4) \times \frac{1}{4} \times 4 = \frac{1}{4} \times 30.$$

$$\therefore \; x = \frac{7}{2}.$$

$$\therefore \text{ volume required} = x \times \frac{1}{4} = \frac{7}{8} \text{ cub. inch.}$$

3. Let the mercury descend x inches, so that the air enclosed now occupies $(1+x)$ inches and is of pressure x inches.

$$\therefore \; (1+x) \times x = 1 \times 30, \text{ by Boyle's Law.}$$

$$\therefore \; x^2 + x - 30 = 0.$$

$$\therefore \; x = 5.$$

4. When the pressure of the enclosed air $= 29 - 28 \cdot 6 = \cdot 4$ ins. the length it occupies $= 33 - 28 \cdot 6 = 4 \cdot 4$.

Let h be the required true height. Then the air occupies $33 - 29 \cdot 48$ ins., *i.e.* $3 \cdot 52$ ins., at a pressure of $h - 29 \cdot 48$.

$$\therefore \; 3 \cdot 52 \times (h - 29 \cdot 48) = \cdot 4 \times 4 \cdot 4, \text{ by Boyle's Law.}$$

$$\therefore \; h = 29 \cdot 48 + \frac{\cdot 4 \times 4 \cdot 4}{3 \cdot 52} = 29 \cdot 48 + \frac{1}{2} = 29 \cdot 98.$$

5. When the pressure of the enclosed air $=28{\cdot}5-27=1{\cdot}5$ ins. of mercury, the length it occupies $=36-27=9$ ins.

Let h be the required true height. Then the air occupies $36-30$, *i.e.* 6 ins., at a pressure of $h-30$ ins.

$$\therefore\ (h-30)\times 6=9\times 1{\cdot}5, \text{ by Boyle's Law.}$$

$$\therefore\ h=30+\frac{4{\cdot}5}{2}=32{\cdot}25.$$

6. If x be the required height, then in the first case the air is of length $x-28$ and of pressure $30-28$, *i.e.* 2 ins. of mercury.

In the second case the length $=x-14{\cdot}6$, and the pressure

$$=15-14{\cdot}6={\cdot}4 \text{ ins.}$$

$$\therefore\ (x-28)\times 2=(x-14{\cdot}6)\times {\cdot}4, \text{ by Boyle's Law.}$$

$$\therefore\ x=31{\cdot}35.$$

7. The difference between the levels of the surfaces of the mercury $=38-4=34$ ins.

Hence the air is subjected to a pressure $=34+29\tfrac{1}{2}=63\tfrac{1}{2}$ ins. of mercury.

Hence, if x be the required length,

$$x\times 29\tfrac{1}{2}=5\times 63\tfrac{1}{2}, \text{ by Boyle's Law.}$$

$$\therefore\ x=\frac{5\times 127}{59}=10\tfrac{45}{59} \text{ ins.}$$

8. Let x be the original length of air in the closed limb, and let h be the height of the mercury barometer. When the mercury is poured in, the difference between the levels of the ends of the mercury is $8-2=6$ and the air is thus at a pressure $6+h$. Hence

$$x\times h=(x-1)\times(6+h)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

When the 11 inches are poured in, the difference $=8+11-4=15$.

$$\therefore\ x\times h=(x-2)\times(15+h)\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Solving, we have $x=6$ and $h=30$.

9. The pressure is that due to $762-700$, *i.e.* 62 mm.

Hence $\quad p=\dfrac{62}{10}\times 13{\cdot}596$ grammes wt. per sq. cm.

$$=84{\cdot}2952 \text{ grammes wt.}$$

10. In the first case the air is of length 1 inch and pressure 1 inch of mercury.

In the second case it is of length $1+31-29\tfrac{1}{2}$, *i.e.* $2\tfrac{1}{2}$ inches, and pressure $h-29\tfrac{1}{2}$, where h is the height required.

$$\therefore\ 2\tfrac{1}{2}\times(h-29\tfrac{1}{2})=1\times 1.$$

$$\therefore\ h=29\tfrac{1}{2}+\frac{2}{5}=29{\cdot}9.$$

11. Let x be the length the air occupies in the barometer, and nx the length it would occupy at atmospheric pressure. Then

$$x \times (30 - 29{\cdot}8) = nx \times 30.$$

$$\therefore \; n = \frac{\cdot 2}{30} = \frac{1}{150}.$$

12. Let the surface of the mercury be lowered y inches. Then the air now occupies $(2+y)$ ins. and its pressure is y ins. of mercury.

$$\therefore \; (2+y) \times y = \frac{1}{2} \times 30.$$

$$\therefore \; y = 3.$$

Let the required true height in the second case be h. Then the air now occupies $(30+2-x)$, *i.e.* $(32-x)$ inches and its pressure is $h-x$ inches.

$$\therefore \; (32-x)(h-x) = \frac{1}{2} \times 30.$$

$$\therefore \; h = x + \frac{15}{32-x}.$$

13. Let x be the length of the air in the first case, *i.e.* at a pressure $\cdot 6$ ins. Then its length in the second case is $x + \cdot 4$ and its pressure $\cdot 4$ ins.

$$\therefore \; x \times \cdot 6 = (x + \cdot 4) \times \cdot 4, \text{ by Boyle's Law,}$$

i.e. $$\cdot 2 \times x = \cdot 16. \quad \therefore \; x = \frac{4}{5} = \cdot 8 \text{ inch.}$$

When the faulty barometer is at 29 the air occupies

$$29{\cdot}8 + \cdot 8 - 29 = 1{\cdot}6 \text{ inch,}$$

and if the true height be then h, the pressure is $h - 29$. Hence

$$1{\cdot}6 \times (h - 29) = \cdot 8 \times \cdot 6, \text{ by Boyle's Law.}$$

$$\therefore \; h - 29 = \frac{\cdot 8 \times \cdot 6}{1{\cdot}6} = \frac{\cdot 6}{2} = \cdot 3.$$

$$\therefore \; h = 29{\cdot}3.$$

14. The length of the air is a inches at a pressure of $(c-b)$ inches.

When the apparent reading is d inches, the length of the air is $a+b-d$ and the pressure is $(x-d)$ inches where x is the height required.

$$\therefore \; (a+b-d) \times (x-d) = a \times (c-b).$$

$$\therefore \; x = d + \frac{a(c-b)}{a+b-d}.$$

EXAMPLES. XXV. (Pages 156—160.)

1. With the notation of Page 154, $a=80$, and $b=6$. Hence $x^2+(80+33\frac{1}{3})x-33\frac{1}{3}\times 6=0$.

$$\therefore\ x=\frac{-170+10\sqrt{307}}{3}.$$

Hence the pressure is equivalent to $\frac{x+a+h}{h}$ atmospheres, *i.e.* to $\frac{17+\sqrt{307}}{10}$ atmospheres, *i.e.* to 3·45 atmospheres.

2. The bell being always kept full of air the depth x of its lowest point is given by

$$\frac{x+h}{h}=\frac{31}{30};\ \therefore\ x=\frac{h}{30}=\frac{1}{30}\times 13\tfrac{1}{2}\times\frac{30}{12}\text{ ft.}=1\tfrac{1}{8}\text{ ft.}$$

3. The rise in the pressure is equivalent to a rise in the water barometer of $13{\cdot}6\times 12\frac{1}{2}\div 12$ feet, *i.e.* to $\frac{85}{6}$ or $14\frac{1}{6}$ feet.

4. Here $b=9$, $h=34$, and $x+a=17$. Hence the equation of Art. 123 gives $x(17+34)=34\times 9$, *i.e.* $x=6$. Hence $a=11$ and required depth $=20$ ft. Let V be the required volume. Thus $V+9\times 25$ cub. ft. at atmospheric press. 34 ft. becomes 9×25 cub. ft. at press. $(34+20)$ ft.

$$\therefore\ (V+225)\times 34=225\times 54.$$

$$\therefore\ V=\frac{20\times 225}{34}=\frac{2250}{17}=132\tfrac{6}{17}\text{ cub. ft.}$$

5. The pressure of the contained air is equivalent to that of $1{\cdot}02\times 100+34$ ft. of fresh water, *i.e.* to 136 ft. Hence, if V be the required volume, then

$$V\times 34=125\times 136.$$

$$\therefore\ V=500\text{ cub. ft.}$$

6. The new pressure is that due to 51 ft. of water, *i.e.* it is $\frac{3}{2}\times$ atmospheric pressure, *i.e.* $\frac{3}{2}\Pi$.

Hence original vol. $\times\,\Pi=$ final volume $\times\frac{3}{2}\Pi$; $\therefore$ etc.

7. With the notation of Art. 123, $x=8$, $b=10$.

$$\therefore\ 64+(a+h)\,.\,8-10h=0.$$

$$\therefore\ 32=h-4a\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

When air has been pumped in, the pressure is that due to $9+a+h$ feet. Hence, if V be the total volume of the bell,

$$\left(V+\frac{67}{440}V\right)\times h=\frac{9}{10}V\times(9+a+h).$$

$$\therefore\ 3a+27=\frac{37}{44}h \qquad\qquad (2).$$

Solving (1) and (2), we have $h=33$, $a=\frac{1}{4}$.

8. The air is reduced to nine-tenths of its original volume; hence the pressure is $\frac{10}{9}$ times the original. Hence, if x be the required height in feet,

$$x\times 13{\cdot}5=\frac{10}{9}\times 33\tfrac{3}{4};\ \therefore x=\frac{25}{9}\text{ ft.}=33\tfrac{1}{3}\text{ ins.}$$

The depth of the surface of the water in the bell $=\left(\frac{10}{9}-1\right)\times$ ht. water barometer $=3\frac{3}{4}$ ft.

9. Let V be the volume of the bell, V_1 the volume of the air at atmospheric pressure that will fill it when its top is at a depth x. Then

$$V_1\,.\,h=V(x+h).$$

$$\therefore\ V_1=V+\frac{V}{h}x.$$

Hence if x increases uniformly it is clear that V_1 increases uniformly also.

10. The pressure of the air inside the bell is that due to a depth $h+nh-\frac{h}{20}$ of water. Hence, if V_1 be the required volume,

$$\frac{4}{5}V\times\left(h+nh-\frac{h}{20}\right)=V_1\times h.$$

$$\therefore\ V_1=\frac{4}{5}\left(n+\frac{19}{20}\right)V.$$

11. By Art. 123, $a=\frac{hb}{x}-h-x$. Hence the question gives

$$\frac{hb}{\frac{2}{3}b}-h-\frac{2}{3}b=3\tfrac{1}{3}\left[\frac{hb}{\frac{3}{4}b}-h-\frac{3}{4}b\right].$$

$$\therefore\ \frac{1}{2}h-\frac{2b}{3}=\frac{10}{3}\left[\frac{1}{3}h-\frac{3}{4}b\right]=\frac{10}{9}h-\frac{5}{2}b.$$

Solving, we have $b=\frac{h}{3}$.

12. The pressure of the air in the diving bell is equal to that at the surface of the water inside of the bell, and is thus greater than that of the water at the level of the top of the bell. Hence when the hole is made the air flows out.

13. The length x of the bell occupied by the air is, as in Art. 123, given by $x^2+81x-340=0$, *i.e.* by $(x+85)(x-4)=0$, and hence $x=4$. The pressure of the contained air is thus that due to $4+47+34$ ft. of water, *i.e.* to 85 ft., and hence its density $=\frac{85}{34}\cdot\sigma=\frac{5}{2}\sigma$.

Let y be the fraction of the wood now immersed in water. Then since the sp. gr. of the wood is $\frac{1}{2}$ we have

$$y\cdot 1+(1-y)\frac{5\sigma}{2}=1\cdot\frac{1}{2}.$$

$$\therefore\ y=\frac{1}{2}\,\frac{1-5\sigma}{1-\frac{5\sigma}{2}}=\frac{1-5\sigma}{2}\left(1+\frac{5\sigma}{2}\right)\text{ approx.}$$

$$=\frac{1}{2}\left(1-\frac{5\sigma}{2}\right)=\frac{2-5\sigma}{4}\text{ approx.}$$

14. The length of the axis now occupied by air is 12 feet, and the volume of the air is therefore $\left(\frac{12}{16}\right)^3$, *i.e.* $\frac{27}{64}$ of its original volume. Hence, if h be the height of the water-barometer, we have

$$\frac{27}{64}(12+33\tfrac{2}{3}+h)=1\cdot h.$$

$$\therefore\ h=33\text{ feet.}$$

15. With the notation of Art. 123, we have $x=\frac{2b}{3}$, and hence the equation of that article gives

$$4b+6\cdot a=3h \qquad\qquad (1).$$

When a volume equivalent to a length $\frac{3b}{2}$ of atmospheric air has been compressed to a length $\frac{b}{2}$, let a' be the new depth of the top of the bell. Then Boyle's Law gives

$$\frac{3b}{2}\cdot h=\frac{b}{2}\left(\frac{b}{2}+a'+h\right).$$

$$\therefore\ a'=2h-\frac{b}{2}.$$

Hence $$a' - a = 2h - \frac{b}{2} - \frac{1}{6}[3h - 4b] = \frac{3h}{2} + \frac{b}{6}.$$

16. The respective heights of the water barometer are $\frac{h\sigma}{12}$ and $\frac{h'\sigma}{12}$ feet, where σ is the sp. gr. of mercury.

(1) If x be the length of the axis immersed,

then $$\frac{h'\sigma}{12} = x + a + \frac{h\sigma}{12},$$

and $$x^3 . \frac{h'\sigma}{12} = b^3 . \frac{h\sigma}{12}.$$

$$\therefore a = \frac{\sigma}{12}(h' - h) - \sqrt[3]{\frac{h}{h'}} \times b.$$

(2) Here $$\frac{h'\sigma}{12} = x + a + \frac{h\sigma}{12},$$

and $$x . \frac{h'\sigma}{12} = b . \frac{h\sigma}{12}.$$

$$\therefore a = (h' - h)\frac{\sigma}{12} - b\frac{h}{h'}.$$

17. The air gets more and more compressed and so displaces less and less water. Finally we must arrive at a point where wt. of vessel + wt. of air = wt. of the water displaced by the vessel and air. If the vessel be then pushed a little further the downward forces are actually greater than the upward forces and the vessel will, if released, sink.

18. The equation of Art. 123 becomes

$$\left(\frac{a}{2}\right)^2 + (h + H)\frac{a}{2} - H . a = 0, \text{ so that } a + 2h = 2H.$$

Let y be the required further distance. Then the water which was of volume a when the pressure was $a + h + H$ becomes of volume $\frac{a}{2}$ when the pressure is $\frac{a}{2} + h + y + H$. Hence

$$a(a + h + H) = \frac{a}{2}\left(\frac{a}{2} + h + y + H\right).$$

$$\therefore y = \frac{3a}{2} + h + H = 4H - 2h.$$

19. The equation of Art. 125 gives $x^2+130x=300$.

$$\therefore\ x+65=\sqrt{4525}.$$

When the temperature is raised let y be the length of the bell occupied by air. Then Art. 114 gives

$$\frac{(y+130)\,y}{1+\frac{15}{273}}=\frac{(x+130)\,x}{1+\frac{20}{273}}.$$

$$\therefore\ y^2+130y=300\times\frac{288}{293}=294{\cdot}88 \text{ nearly}.$$

$$\therefore\ y+65=\sqrt{4519{\cdot}88}.$$

Now by Art. 123 the tensions of the chain are

$$W-\pi\,.\,9\,.\,xw \quad\text{and}\quad W-\pi\,.\,9\,.\,y\,.\,w.$$

Hence increase in the tension

$$=\pi\,.\,9\,.\,w\,(x-y)=\pi\,.\,9\,.\,62\tfrac{1}{2}\,[\sqrt{4525}-\sqrt{4519{\cdot}88}]$$

$$=\pi\,.\,9\,.\,62\tfrac{1}{2}\,.\,\frac{4525-4519{\cdot}88}{\sqrt{4525}+\sqrt{4519{\cdot}88}}=\pi\times 562\tfrac{1}{2}\times\frac{5{\cdot}12}{2\sqrt{4520}} \text{ nearly}$$

$$=\frac{22}{7}\times\frac{1125}{2}\times\frac{5{\cdot}12}{2\times 67{\cdot}2}=\frac{126720}{1881{\cdot}6} \text{ nearly}=67 \text{ lbs. wt. nearly.}$$

21. Initial tension $=W_1-A\,.\,w\,.\,x.$

Let V be the volume of the water drawn up so that $Vw=W$. Then, if y be the new length of the bell occupied by the air, its new volume is $Ay-V$, and thus

$$(Ay-V)\,(y+a+h)=Ax\,(x+a+h).$$

$$\therefore\ (y-x)\,.\,A\,.\,(y+x+a+h)=V\,(y+a+h),$$

$$y-x=\frac{V}{A}\,\frac{y+a+h}{y+x+a+h}.$$

Since V is small y and x are very nearly equal and the right-hand side of this equation $=\dfrac{V}{A}\cdot\dfrac{x+a+h}{2x+a+h}$ approx.

Hence increase of tension

$$=(W_1+W-Ayw)-(W_1-Axw)$$

$$=W-A\,(y-x)\,w=W-Vw\,\frac{x+a+h}{2x+a+h}$$

$$=W-W\,\frac{x+a+h}{2x+a+h}=W\,\frac{x}{2x+a+h}.$$

22. If A be the internal section of the bell, then

$$Aaw = W.$$

If x be the original length occupied by the air, then

$$x^2 + (d+h)\,x = ha \quad\text{..........................(1).}$$

When the temperature is raised let y be the final length occupied by the air. Then Art. 114 gives

$$\frac{y\,(y+d+h)}{1+at_1} = \frac{x\,(x+d+h)}{1+at} = \frac{ha}{1+at}.$$

$$\therefore\ y^2 + (d+h)\,y = ha\,\frac{1+at_1}{1+at} \quad\text{......................(2).}$$

Now final tension – original tension

$$= W - Awy - (W - Awx) = -Aw\,(y-x) \quad\text{............ (3).}$$

Also, from (1) and (2), by subtraction

$$(y-x)\,(y+x+d+h) = haa\,\frac{t_1 - t}{1+at}.$$

$$\therefore\ y - x = haa\,\frac{t_1-t}{1+at}\,\frac{1}{y+x+d+h}$$

$$= haa\,\frac{t_1-t}{1+at}\,\frac{1}{2x+d+h}\ \text{nearly.}$$

Therefore final tension – original tension

$$= -Wha\,.\,\frac{t_1-t}{1+at}\,\frac{1}{\sqrt{(h+d)^2+4ah}}\ \text{from equation (1).}$$

23. The volume now occupied by the air is to the original volume as x^3 to a^3. Also the new pressure is that due to a depth $x+d+h$ of water. Hence by Boyle's Law

$$x^3\,(x+d+h) = a^3\,.\,h \quad\text{........................ (1).}$$

Let y be the length of the part of the bell occupied by the air when the temperature is raised. The relation of Art. 114 then gives

$$\frac{y^3\,(y+d+h)}{1+a\,(T+t)} = \frac{x^3\,(x+d+h)}{1+aT} = \frac{a^3h}{1+aT}.$$

$$\therefore\ y^4 + (d+h)\,y^3 = \left(1 + \frac{at}{1+aT}\right) a^3h \quad\text{.............. (2).}$$

Subtracting (1) from (2), we have

$$y^4 - x^4 + (d+h)\,(y^3 - x^3) = \frac{at}{1+aT}\,a^3h.$$

$$\therefore\ y - x = \frac{a^3h}{(y^2+x^2)\,(y+x) + (d+h)\,(y^2+xy+x^2)} \times \frac{at}{1+aT}.$$

Thus since a is small, y and x are very nearly equal and the right-hand member of this equation

$$=\frac{a^3h}{4x^3+3x^2(d+h)}\cdot\frac{at}{1+aT}\text{ approx.}$$

Hence diminution in the tension

$$=\left(W_1-V\frac{x^3}{a^3}w\right)-\left(W_1-V\frac{y^3}{a^3}w\right),$$

where V is the volume of the cone,

$$=\frac{Vw}{a^3}(y^3-x^3)$$

$$=\frac{W}{a^3}(y-x)(y^2+xy+x^2)$$

$$=\frac{W}{a^3}(y-x)\times 3x^2\text{ nearly,}$$

$$=\frac{W}{a^3}\times 3x^2\times\frac{a^3h}{4x^3+3x^2(d+h)}\,\frac{at}{1+aT}$$

$$=\frac{3Wh}{4x+3d+3h}\cdot at,\text{ neglecting squares of }a.$$

24. x being the length occupied by air originally, we have

$$x^2+(a+h)x=hb\ \dots\dots\dots\dots\dots\dots(1).$$

If h_1 be the new height of the water-barometer we have, if A be the area of the cross-section of the bell,

$$P=A(h_1-h)w,$$

so that

$$h_1=h+\frac{P}{Aw}.$$

If $x+\xi$ be the new length occupied by air, we have similarly

$$(x+\xi)^2+(a+h_1)(x+\xi)=h_1b\ \dots\dots\dots\dots\dots\dots(2).$$

Subtracting (1) from (2), we have

$$2x\xi+\xi^2+x\cdot\frac{P}{Aw}+\xi\left(a+h+\frac{P}{Aw}\right)=b\cdot\frac{P}{Aw}.$$

Now P is small, and so also must ξ be; hence neglecting ξ^2 and ξP, this equation gives

$$\xi[2x+a+h]=\frac{P}{Aw}(b-x).$$

Therefore increase in the tension

$$=W_1-Aw(x+\xi)-[W_1-Awx]$$

$$=-Aw\xi=-P\frac{b-x}{2x+a+h}$$

$$=\frac{P}{2}\left[1-\frac{a+h+2b}{2x+a+h}\right]$$

$=$ given answer, by equation (1).

EXAMPLES. XXVI. (Pages 172—174.)

1. Original height $=13{\cdot}6\times28$ ins. $=31{\cdot}7\dot{3}$ ft.

   Final height $=13{\cdot}6\times31$ ins. $=35{\cdot}1\dot{3}$ ft.

2. The height of the petroleum-barometer would

$$=\frac{33 \text{ ft. } 8 \text{ ins.}}{{\cdot}8}=42 \text{ ft. } 1 \text{ inch.} \quad \therefore \text{ etc.}$$

3. Height of the sea-water barometer

$$=\frac{34 \text{ ft. } 2 \text{ ins.}}{1{\cdot}025}=\frac{40}{41}\times34\tfrac{1}{6}\text{ ft.}=\frac{200}{6}\text{ ft.}=33 \text{ ft. } 4 \text{ ins.}$$

4. If A be the area of the section of the barrel in sq. ft., then

$$A\,.\,1\,.\,w=10.$$

By Art. 130 the tension of the piston rod

$$=A\,.\,w\,.\,24.$$

$$\therefore \text{ work done}=A\,.\,w\,.\,24\times\frac{1}{3}\text{ ft.-lbs.}$$

$$=10\times\frac{24}{3}\text{ ft.-lbs.}=80 \text{ ft.-lbs.}$$

5. The air which at atmospheric pressure occupied ${\cdot}4$ ins. will at end of the stroke occupy 6 ins. and its pressure then $=\frac{{\cdot}4}{6}\times$ atmospheric press. which is equivalent to $\frac{{\cdot}4}{6}\times32$ ft. of water, *i.e.* to $2\frac{2}{15}$ ft.

The water would therefore rise in the barrel to a height finally of $(32-2\frac{2}{15})$ ft., *i.e.* to $29\frac{13}{15}$ ft. and would therefore finally reach the pump barrel.

6. With the notation of Art. 131 we have $c=16$, $A=16a$, $h=32$; hence the height x_1 at the end of the first stroke is given by

$$32\times 16=(32-x_1)\,[16-x_1+16l].$$

If $x_1=16$, this gives $32=16l$, *i.e.* $l=2$ ft.

If however we are given $l=1$, we have

$$32\times 16=(32-x_1)\,(16-x_1+16)=(32-x_1)^2.$$
$$\therefore\ 16\sqrt{2}=32-x_1.$$
$$\therefore\ x_1=32-16\sqrt{2}=9\cdot 37\ \ldots\ \text{ft.}$$

7. Tension $=$ wt. of $\dfrac{100}{144}\times 200$ cubic ft. of water

$$=\frac{100\times 200}{144}\times\frac{125}{2}\ \text{lbs. wt.}=8680\tfrac{5}{9}\ \text{lbs. wt.}$$

8. Force $=$ wt. of $\dfrac{10}{144}\times 60$ cub. ft. of water

$$=\frac{600}{144}\times\frac{125}{2}\ \text{lbs. wt.}=260\tfrac{5}{12}\ \text{lbs. wt.}$$

9. Force to raise the piston $=$ wt. of $\dfrac{\pi\,.\,6^2}{4\times 144}\times 20$ cub. ft. of water

$$=\frac{5\,.\,\pi}{4}\times\frac{125}{2}\ \text{lbs. wt.}=\frac{625\pi}{8}\ \text{lbs. wt.}$$

Force to depress the piston

$$=\text{wt. of }\frac{\pi}{4}\,.\,\frac{6^2}{144}\times 100\ \text{cub. ft. of water}$$
$$=\frac{3125\pi}{8}\ \text{lbs. wt.}$$

10. By Art. 130 the tension of the piston rod when the water is at $L=A\,.\,w\,.\,CL$.

Therefore work done per stroke $=$ tension $\times\,BL$

$$=A\,.\,w\,.\,CL\,.\,BL.$$

11. Force in the backward stroke

$$=\text{wt. of }\pi\,.\,10^2\,.\,4\,.\,100\ \text{cub. cms. of water}$$
$$=40\pi\ \text{kilogrammes' wt.}$$

Force in the forward stroke

$$=\text{wt. of }\pi\,.\,10^2\,.\,60\,.\,100\ \text{cub. cms.}$$
$$=600\pi\ \text{kilogrammes' wt.}$$

12. With the notation of Art. 131, $A=a$, and

$$\therefore\ hc=(h-x_1)(c-x_1+l),$$

and $$(h-x_1)(c-x_1)=(h-x_2)(c-x_2+l).$$

Also we are given that $x_2=2x_1$.

$$\therefore\ x_1^2-x_1(h+c+l)+hl=0\ \dots\dots\dots(1),$$

and $$3x_1^2-x_1(h+c+2l)+hl=0\ \dots\dots\dots(2).$$

Subtracting, we have $x_1=\frac{l}{2}$, and hence (1) gives $l=2h-2c$.

Therefore $2h=c+(c+l)=$ sum of the greatest and least distances of the piston from the surface of the water in the well.

13. With the notation of Art. 131, we have

$$A=5a,\quad h=34,\quad c=10\ \text{ and }\ x_1=c=10.$$

Therefore the first equation at the bottom of Page 167 gives

$$34\times 10=(34-10)[10-10+5l].$$

$$\therefore\ l=\frac{34\times 2}{24}=2\tfrac{5}{6}\text{ ft.}$$

14. At the beginning of the stroke the air under the piston is at atmospheric pressure and of length 3 inches. At the end of the stroke its length is 12 inches, and its pressure $=\frac{3}{12}\times$ atmospheric pressure $=$ a pressure of $\frac{1}{4}\times 34$ ft. of water $=$ pressure of $8\frac{1}{2}$ ft. of water.

Hence the greatest height to which the water will rise

$$=34-8\tfrac{1}{2}=25\tfrac{1}{2}\text{ ft.}$$

15. With the notation of Art. 131 we have

$$ch=(h-x_1)[c-x_1+nl]\ \dots\dots\dots(1),$$

and $$(h-x_1)(c-x_1)=(h-x_2)[c-x_2+nl]\ \dots\dots\dots(2).$$

Also we are given that $x_2=c$.

$$\therefore\ (h-x_1)(c-x_1)=nl(h-c).$$

$\therefore$ (1) gives $$ch=nl(h-x_1)+nl(h-c).$$

$$\therefore\ x_1=2h-c-\frac{ch}{nl}.$$

Substituting in (1), we have

$$ch=\left[c+\frac{ch}{nl}-h\right]\left[2c-2h+\frac{ch}{nl}+nl\right]$$

$$=c(2c+nl)-h\left[3c+nl-\frac{3c^2}{nl}\right]+h^2\left[1-\frac{c}{nl}\right]\left[2-\frac{c}{nl}\right].$$

EXAMPLES. XXVII. (Pages 183—185.)

1. Here $$\left[\frac{V}{V+V'}\right]^4 \rho = \frac{81}{256}\rho = \left(\frac{3}{4}\right)^4 \rho.$$
$$\therefore\ V = 3V'.$$

2. $V=36.\quad V'=1\times 4=4.\quad \therefore\ \dfrac{V}{V+V'}=\dfrac{9}{10}.$
$$\therefore \text{ required ratio of pressures} = \left(\frac{9}{10}\right)^{\text{[illegible]}}.$$

3. In the first case $$\frac{V}{V+V'} = \frac{10}{10+1} = \frac{10}{11},$$
and in the second $$\frac{V}{V+V'} = \frac{5}{5+1} = \frac{5}{6}.$$
Hence required ratio $= \left(\dfrac{10}{11}\right)^3 : \left(\dfrac{5}{6}\right)^3$
$$= 12^3 : 11^3 = 1728 : 1331.$$

4. $\dfrac{V}{V+V'} = \dfrac{10}{10+1} = \dfrac{10}{11}.$ Hence required ratio of pressures
$$= 10^8 : 11^8 = 100000000 : 214358881.$$

5. $$\frac{V}{V+V'} = \frac{6}{7}. \qquad \text{Now } \left(\frac{6}{7}\right)^4 = \frac{1296}{2401} > \frac{1}{2},$$
$$\left(\frac{6}{7}\right)^5 = \frac{7776}{16807} < \frac{1}{2}; \qquad \left(\frac{6}{7}\right)^6 = \frac{46656}{117649} > \frac{1}{3},$$
$$\left(\frac{6}{7}\right)^7 = \frac{279936}{823543} > \frac{1}{3}; \qquad \left(\frac{6}{7}\right)^8 = \frac{1679616}{5764801} < \frac{1}{3}.$$
$\therefore$ etc.

6. In the first case we have $$\frac{V}{V+V'} = \frac{12}{13},$$
and in the second $$\frac{V}{V+V'} = \frac{6}{7}.$$
Hence we want x where $\left(\dfrac{6}{7}\right)^x = \left(\dfrac{12}{13}\right)^6 = \dfrac{2985984}{4826809}.$

Also $\left(\dfrac{6}{7}\right)^3 = \dfrac{216}{343};\ \left(\dfrac{6}{7}\right)^4 = \dfrac{1296}{2401}$ which is $< \left(\dfrac{12}{13}\right)^6.$

More easily $$x=6\times\frac{\log 12-\log 13}{\log 6-\log 7}$$

$$=6\times\frac{1{\cdot}1149434-1{\cdot}0791812}{{\cdot}8450980-{\cdot}7781513}=3{\cdot}2\ldots \text{ on reduction.}$$

Hence the answer is 4 strokes.

7. Here $$\left(\frac{V}{V+V'}\right)^{10}=\frac{20}{30}=\frac{2}{3}.$$

$$\therefore\ \left(\frac{V}{V+V'}\right)^{30}=\left(\frac{2}{3}\right)^3=\frac{8}{27}.$$

$\therefore$ height of mercury required$=\frac{8}{27}\times 30$ inches$=8\frac{8}{9}$ inches.

8. With the notation of Art. 141, $h=6$, $a=b=\frac{1}{4}$.

$$\therefore\ \frac{ab}{(h+a)(h+b)}=\left(\frac{a}{h+a}\right)^2=\left(\frac{\frac{1}{4}}{6+\frac{1}{4}}\right)^2=\frac{1}{625}.$$

9. The required number is x, where

$$\frac{1000+x\,.\,80}{1000}=4.\quad \therefore\ x=37\tfrac{1}{2}.$$

10. $$V=\pi\,.\,\frac{1}{2^2}\,.\,80\,;\quad V'=\pi\,.\,\frac{1}{2^2}\,.\,8.$$

Hence, if x be the required number of strokes,

$$2=\frac{V+xV'}{V}=1+x\,.\,\frac{1}{10}.\quad \therefore\ x=10.$$

11. If p be the greatest pressure that can be exerted by the compressed air, then $165=5\,(p-15)$. $\therefore\ p=48$.

Hence, if x be the required number of strokes,

$$\frac{10+x\,.\,1}{10}=\frac{48}{15}=\frac{16}{5}.\quad \therefore\ x=22.$$

12. A volume B of air at atmospheric pressure is condensed into a volume $B-C$, and its pressure becomes therefore $\frac{B}{B-C}$ atmospheres. When the air inside the receiver is at this or a greater pressure the valve in the receiver will thus not be opened.

13. Here $V=8V'$, so that $\frac{V}{V+V'}=\frac{8}{9}$, so that the density of the air in the receiver at the end of the fourth stroke is $\left(\frac{8}{9}\right)^4 . \rho$. This is also the density in the barrel at the beginning of the fifth stroke. If the upper valve open when a fraction x of the barrel has been described by the piston, then a volume unity at pressure $\left(\frac{8}{9}\right)^4 \rho$ has become $1-x$ at pressure ρ, the weight of the upper valve being neglected.

$$\therefore \left(\frac{8}{9}\right)^4 \rho . 1=\rho . (1-x), \text{ by Boyle's Law.}$$

$$\therefore x=1-\left(\frac{8}{9}\right)^4=\frac{2465}{6561}.$$

14. Let ρ be the density of atmospheric air and ρ' the density required. Then unit volume of density ρ' has become $\frac{1}{4}$ unit volume at density ρ. $\therefore \rho'=\frac{1}{4}\rho$, by Boyle's Law.

15. Let V be the volume of the speaking tube between the open end and the obstruction. Then $V'=50$, and we have

$$4=\frac{V+30V'}{V}=\frac{V+30\times 50}{V}.$$

$$\therefore V=500.$$

Also the section of the tube is one square inch; hence length required

$$=500 \text{ inches}$$

$$=41\tfrac{2}{3} \text{ feet}.$$

16. $V=20V'$. The density at the end of 20 strokes of the condenser $=\frac{V+20V'}{V} . \rho=2\rho$.

Also $$\frac{V}{V+V'}=\frac{20}{21}.$$

Therefore density at end of 14 strokes more $=\left(\frac{20}{21}\right)^{14}\times 2\rho$, and it can easily be shewn that $\left(\frac{20}{21}\right)^{14}=\frac{1}{2}$ nearly.

17. When the piston is at its highest point let σ be the density in the barrel. As the piston moves down, the air of volume C expands to volume $B-C'$ and hence its least density is $\frac{C . \sigma}{B-C'}$. Hence, if ρ

be the density of the air in A, then for the valve from A to open we must have

$$\rho > \sigma . \frac{C}{B - C'} \text{..........................(1)}.$$

Similarly the air of volume $B - C$ below the piston becomes compressed into a volume C' and hence its greatest density is $\dfrac{\sigma . (B - C)}{C'}$. If ρ' be the density in the receiver then, for the valve in it to open, we must have

$$\sigma . \frac{B - C}{C'} > \rho' \text{..........................(2)}.$$

From (1) and (2), in order that the machine may work,

$$\rho > \frac{CC'}{(B - C)(B - C')} \rho'.$$

Hence etc.

18. Let x be the maximum pressure of the air in the condenser. When the piston is at the lowest point there is a volume v' beneath it of pressure $x + p$, since the pressure on the condenser-valve is just sufficient not to open it. When the piston is at its highest point the pressure of this air is $\dfrac{(x+p)\,v'}{v}$, by Boyle's Law. Since the atmospheric air can now just not open the piston-valve, this pressure must be $\Pi - p$.

$$\therefore\ (x+p)\frac{v'}{v} = \Pi - p.$$

$$\therefore\ x = \frac{v}{v'}(\Pi - p) - p.$$

19. At the end of the $(n-1)$th stroke let ρ_{n-1} be the density in the receiver and barrel, and ρ the original atmospheric density. At the end of the next half stroke there is beneath the piston a volume A of density ρ_{n-1}, C of density ρ, and this at the end of the nth stroke becomes $(A + B)$ of density ρ_n.

$$\therefore\ A\rho_{n-1} + C . \rho = (A + B)\,\rho_n, \text{ by Boyle's Law.}$$

Hence
$$\rho_n - \frac{C}{B}\rho = \frac{A}{A+B}\rho_{n-1} + \frac{C\rho}{A+B} - \frac{C\rho}{B}$$
$$= \frac{A}{A+B}\rho_{n-1} - \frac{AC}{B(A+B)}\rho$$
$$= \frac{A}{A+B}\left(\rho_{n-1} - \frac{C}{B}\rho\right).$$

So
$$\rho_{n-1} - \frac{C}{B}\rho = \frac{A}{A+B}\left(\rho_{n-2} - \frac{C}{B}\rho\right),$$

....

$$\rho_1-\frac{C}{B}\rho=\frac{A}{A+B}\left(\rho-\frac{C}{B}\rho\right).$$

Hence, by multiplication and cancelling of like terms,

$$\rho_n-\frac{C}{B}\rho=\left(\frac{A}{A+B}\right)^n\left(\rho-\frac{C}{B}\rho\right).$$

$$\therefore\ \frac{\rho_n}{\rho}=\frac{C}{B}+\left(1-\frac{C}{B}\right)\left(\frac{A}{A+B}\right)^n.$$

EXAMPLES. XXVIII. (Page 189.)

1. Ans. $=13\cdot6\times30$ ins. $=34$ ft.

2. Ans. $=\dfrac{30\times13\cdot6}{1\cdot5}$ ins. $=272$ ins. $=22$ ft. 8 ins.

3. He can only make his siphon work for a height equal to that of the mercury-barometer, and this height is much less than 36 inches.

4. Let the siphon just cease working when the depth of the surface of the water below the top of the cistern is x. If h be the height of the cylinder, or water-barometer, the pressure of this air then

$$=\frac{\Pi\,.\,\frac{h}{4}}{x}=wh\,.\,\frac{h}{4x},$$

and the pressure at the bottom of the water

$$=\frac{wh^2}{4x}+w\,(h-x).$$

This must just equal wh. Hence

$$\frac{h^2}{4x}-x=0.$$

$$\therefore\ x=\frac{h}{2},$$

and depth by which the surface of the water is lowered

$$=\frac{h}{2}-\frac{h}{4}=\frac{h}{4}.$$

Hence one-third of the water is removed.

5. If a hole be made in the shorter limb BC the flow ceases, the fluid below the hole falling back into the vessel.

If it be made in the longer arm below the level of Q the motion goes on, the liquid flowing out at this hole. If it be made in the longer arm above Q, the action of the instrument stops and the liquid flows back into the vessel.

EXAMPLES. XXIX. (Pages 199, 200.)

1. With the Figure of Page 193, if $B'PC'$ be the horizontal line through P and G' the centre of gravity of the $\triangle$ cut off, then

$$DG'=DP+\frac{1}{3}PA=\frac{1}{2}DA+\frac{1}{6}DA=\frac{2}{3}DA.$$

$$\therefore \frac{\text{thrust on } \triangle\, AB'C'}{\text{thrust on } \triangle\, ABC}=\frac{\text{area of } AB'C' \times \text{depth of } G'}{\text{area of } ABC \times \text{depth of its C.G.}}$$

$$=\frac{AP^2\times\frac{2}{3}\text{ depth of } A}{AD^2\times\frac{1}{3}\text{ depth of } A}=\frac{1\times 2}{4\times 1}=\frac{1}{2}. \quad \therefore \text{ etc.}$$

2. Let $ABCD$ be the symmetrical section of the box, AB being the lid, and ABC the string whose tension is T.

Taking moments about A, we have

$$T\,.\,AB=AB^2\times\frac{AB}{2}w\times\text{depth of centre of pressure of the lid below } A.$$

$$\therefore\ T=\frac{w}{2}\,.\,AB^2\times\frac{2}{3}AB=\frac{1}{3}w\,.\,AB^3=\text{etc.}$$

3. Let a, a, $\frac{1}{2}a$ be the lengths of the edges of the box and w_1 its wt. per sq. foot. Taking moments about the hinged edge, we have

$$w_1\,.\,a^2\,.\,\frac{a}{2}+4\,.\,w_1\frac{a^2}{2}\,.\,\frac{a}{4}$$

$=$ moment of the pressure on the outside face
$-$ moment of the weight of the water

$$=a^2\,.\,\frac{a}{2}w\,.\,\frac{2a}{3}-w\,.\,\frac{1}{2}a^3\,.\,\frac{a}{4},$$

i.e.
$$w_1a^3=wa^4\left[\frac{1}{3}-\frac{1}{8}\right]=\frac{5}{24}wa^4.$$

$$\therefore\ w_1=\frac{5}{24}wa.$$

4. If c be the horizontal side, we have a thrust $wbc\,.\,\frac{b}{2}$ at a depth $\frac{2b}{3}$ and $-wac\,.\,\frac{a}{2}$ at a depth $\frac{2a}{3}$. Hence required depth

$$=\frac{\frac{wb^2c}{2}\,.\,\frac{2b}{3}-w\frac{a^2c}{2}\,.\,\frac{2a}{3}}{\frac{wb^2c}{2}-w\frac{a^2c}{2}}=\frac{2}{3}\frac{b^3-a^3}{b^2-a^2}=\frac{2}{3}\frac{b^2+ba+a^2}{b+a}.$$

5. Let $ABCD$ be the trapezium, AD being the side in the water. Draw AE, DF $\perp^{r}$ to BC.

The thrust on $AEFD = ahw \cdot \frac{h}{2}$ at a depth $\frac{2h}{3}$.

The thrust on $AEB = \frac{1}{2} \cdot BE \cdot hw \cdot \frac{2h}{3}$ at a depth $\frac{3h}{4}$.

That on $DFC = \frac{1}{2} \cdot FC \cdot hw \cdot \frac{2h}{3}$ at a depth $\frac{3h}{4}$.

$$\therefore \text{ required depth} = \frac{\frac{1}{2} ah^2 w \times \frac{2h}{3} + \frac{wh^2 \cdot BE}{3} \times \frac{3h}{4} + \frac{wh^2 \cdot FC}{3} \times \frac{3h}{4}}{\frac{1}{2} ah^2 w + \frac{wh^2 \cdot BE}{3} + \frac{wh^2 \cdot FC}{3}}$$

$$= h \frac{a + \frac{BE + FC}{4}}{\frac{1}{2} a + \frac{BE + FC}{3}} = h \frac{4a + 3(b-a)}{6a + 4(b-a)} = \frac{h}{2} \cdot \frac{a + 3b}{a + 2b}.$$

6. Produce BA to meet CD produced in E, so that

$$DE = \frac{\alpha}{\beta - \alpha} h = h',$$

where

$$CD = h.$$

The thrust on $ABCD$ is the resultant of the thrust on EBC and a thrust equal and opposite to that on ADE.

The former $= \frac{1}{2} \beta (h + h') \cdot \frac{\beta}{3}$ and acts at a depth $\frac{\beta}{2}$.

The latter $= \frac{1}{2} \alpha h' \cdot \frac{\alpha}{3}$ and acts at a depth $\frac{\alpha}{2}$.

$$\therefore \text{ required depth} = \frac{\frac{\beta^2}{6} (h + h') \times \frac{\beta}{2} - \frac{\alpha^2}{6} h' \times \frac{\alpha}{2}}{\frac{\beta^2}{6} (h + h') - \frac{\alpha^2}{6} h'}$$

$$-\frac{1}{2} \frac{\beta^3 \cdot h \cdot \frac{\beta}{\beta - \alpha} - \alpha^3 \cdot \frac{\alpha}{\beta - \alpha}}{\beta^2 \cdot h \cdot \frac{\beta}{\beta - \alpha} - \alpha^2 \cdot \frac{\alpha}{\beta - \alpha}} = \frac{1}{2} \frac{\beta^4 - \alpha^4}{\beta^3 - \alpha^3} = \text{etc.}$$

7. Let $ABCD$ be the lid, AB being the line of the hinges. In each case the moment of the weight W of the lid about AB must just equal the moment of the thrust on the lid. Let the required angles be θ_1, θ_2, θ_3 when the box is turned about the edges $\parallel^1$ to AB, CD, BC respectively. Then, if a be the length of an edge,

$$W \cdot \frac{a}{2} \cos \theta_1 = wa^2 \cdot \frac{a}{2} \sin \theta_1 \cdot \frac{a}{3} \quad \text{(1)},$$

$$W \cdot \frac{a}{2} \cos \theta_2 = wa^2 \cdot \frac{a}{2} \sin \theta_2 \cdot \frac{2a}{3} \quad \text{(2)},$$

and

$$W \cos \theta_3 \cdot \frac{a}{2} = wa^2 \cdot \frac{a}{2} \sin \theta_3 \cdot \frac{a}{2} \quad \text{(3)}.$$

$$\therefore \cot \theta_1 : \cot \theta_2 : \cot \theta_3 :: \frac{1}{6} : \frac{2}{6} : \frac{1}{4}.$$

$$\therefore \tan \theta_1 : \tan \theta_2 : \tan \theta_3 :: 6 : 3 : 4.$$

8. In the figure of Art. 154 draw $CK \parallel^1$ to BA to meet the surface in K. Let $AK = BC = a$, and let $\perp^r$ from A on $BC = h$. Then the thrust on ABC is the resultant of the thrust on $ABCK$ and a thrust equal and opposite to that on ACK.

The thrust on $ABCK \propto ah \times \frac{h}{2}$, and acts at a dist. $\frac{2h}{3}$ from AK.

The thrust on $ACK \propto \frac{1}{2} ah \times \frac{h}{3}$, and acts at a dist. $\frac{h}{2}$ from AK.

$$\therefore \text{ required dist.} = \frac{\frac{ah^2}{2} \times \frac{2h}{3} - \frac{ah^2}{6} \times \frac{h}{2}}{\frac{ah^2}{2} - \frac{ah^2}{6}}$$

$$= h \frac{\frac{1}{3} - \frac{1}{12}}{\frac{1}{2} - \frac{1}{6}} = h \frac{4-1}{6-2} = \frac{3h}{4}.$$

EXAMPLES. XXX. (Pages 203, 204.)

1. Originally the thrust was $a^2 \cdot \frac{a}{2} w$ at a depth $\frac{2a}{3}$ below the highest side of the square; when the latter is lowered an additional thrust $a^2 \cdot bw$ is caused at a depth $\frac{a}{2}$.

Therefore depth of new centre of pressure below the highest side

$$=\frac{\frac{a}{2}w\,.\,\frac{2a}{3}+bw\,.\,\frac{a}{2}}{\frac{a}{2}w+bw}=\frac{4a^2+6ab}{6\,(a+2b)}=\frac{a}{2}+\frac{a^2}{6\,(a+2b)}.\quad \therefore \text{ etc.}$$

2. We have to find the centre of pressure of two thrusts; one equal to $\frac{1}{2}wah\,.\,\frac{2h}{3}$ acting at a depth $k-\frac{1}{4}h$, and the other equal to $\frac{1}{2}wah\,.\,(k-h)$ acting at a depth $k-\frac{h}{3}$.

$$\therefore \text{ required depth}=\frac{\frac{2h}{3}\left(k-\frac{1}{4}h\right)+(k-h)\left(k-\frac{h}{3}\right)}{\frac{2h}{3}+k-h}=\text{etc.}$$

3. Here, as in the last example, we have to find the resultant of two thrusts one of which is that which would act on the triangle if its upper side were in the surface of the liquid, and the other due to the superincumbent liquid of depth $k-h$. These thrusts are

$$\frac{1}{2}wah\,.\,\frac{h}{3}\text{ acting at a depth }k-\frac{h}{2},$$

and $$\frac{1}{2}wah\,.\,(k-h)\text{ acting at a depth }k-\frac{2h}{3}.$$

$$\therefore \text{ required depth}=\frac{\frac{h}{3}\left(k-\frac{h}{2}\right)+(k-h)\left(k-\frac{2h}{3}\right)}{\frac{h}{3}+k-h}=\text{etc.}$$

4. If a be the area of the $\triangle$ we have two thrusts; one,

$$aw\,.\,\frac{h}{3}\text{ acting at a depth }\frac{h}{2};$$

the other, $$aw\,.\,34\text{ acting at a depth }\frac{h}{3}.$$

Also $$h=\text{height of the }\triangle=6\sqrt{3}\,.\,\sin 60^\circ=9.$$

$$\therefore \text{ required depth}=\frac{\frac{h}{3}\times\frac{h}{2}+34\times\frac{h}{3}}{\frac{h}{3}+34}=\frac{\frac{27}{2}+102}{37}=\frac{231}{74}=3\tfrac{9}{74}\text{ ft.}$$

5. Let a be the area and k the vertical height of the triangle. When the atmospheric pressure is neglected, the thrust is $aw\frac{k}{3}$ at a depth $\frac{k}{2}$.

The atmospheric pressure gives a thrust awh at a depth $\frac{k}{3}$.

Therefore depth of new centre of pressure

$$=\frac{\frac{k}{3}\cdot\frac{k}{2}+h\cdot\frac{k}{3}}{\frac{k}{3}+h}=\frac{1}{2}\frac{k(k+2h)}{k+3h}$$

$$=\frac{k}{2}-\frac{1}{2}\frac{kh}{k+3h}=\text{original depth}-\frac{1}{2}\frac{h\delta}{\delta+h},$$

since $$k=3\delta.$$

6. We here want the centre of pressure of a thrust $aw\,.\,\frac{2k}{3}$ at a depth $\frac{3k}{4}$, and of a thrust $aw\,.\,h$ at a depth $\frac{2k}{3}$.

$$\therefore\ \text{depth}=\frac{\frac{2k}{3}\cdot\frac{3k}{4}+h\cdot\frac{2k}{3}}{\frac{2k}{3}+h}=\frac{6k^2+8hk}{4(2k+3h)}$$

$$=\frac{3k}{4}-\frac{hk}{4(2k+3h)}=\frac{3k}{4}-\frac{h\delta}{8(\delta+h)},$$

since $\delta=\frac{2k}{3}$ in this case. $\therefore$ etc.

7. Let H be the height of the water-barometer, P the centre of pressure when atmospheric pressure is neglected, P' when it is taken into account, and G the centre of gravity. Let h, k be the depths of G and P. Then P' is the centre of action of thrusts wah at P and of waH at G.

Hence P' lies on PG and its depth

$$=\frac{h\,.\,k+H\,.\,h}{h+H}=h\frac{k+H}{h+H},\quad i.e.\ \frac{\text{depth of }P'}{\text{depth of }G}=\frac{k+H}{h+H}.\quad\therefore\ \text{etc.}$$

8. As in Art. 159, the distance between the centres of mass and pressure

$$=\frac{ab-a^2}{h+a}=\frac{\lambda}{a+vt}$$

at any time t, where λ is a constant and v is the given velocity.

Therefore rate at which the two centres approach

$$= \underset{\tau=0}{\mathrm{Lt}} \left[\frac{\lambda}{a+v(t+\tau)} - \frac{\lambda}{a+vt}\right] \div \tau$$

$$= \underset{\tau=0}{\mathrm{Lt}} \left[\frac{-\lambda v\tau}{[a+vt][a+v(t+\tau)]} \div \tau\right]$$

$$= -\frac{\lambda v}{(a+vt)^2},$$

and it thus varies inversely as $(a+vt)^2$; *i.e.* as the square of the depth of the centre of mass.

EXAMPLES. XXXI. (Pages 208—212.)

1. By Art. 162, depth of C.P.

$$= \frac{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta}{2(\alpha+\beta+\gamma)}.$$

Also depth of C.G. = that of equal particles at the angular points

$$= \frac{\alpha+\beta+\gamma}{3}.$$

$$\therefore \text{ excess} = \frac{\alpha^2+\beta^2+\gamma^2+\beta\gamma+\gamma\alpha+\alpha\beta}{2(\alpha+\beta+\gamma)} - \frac{\alpha+\beta+\gamma}{3} = \text{etc.}$$

The depth of the centre of the given parallel forces

$$= \frac{\alpha(2\alpha+\beta+\gamma)+\beta(\alpha+2\beta+\gamma)+\gamma(\alpha+\beta+2\gamma)}{(2\alpha+\beta+\gamma)+\ldots+\ldots}$$

$$= \frac{2\alpha^2+2\beta^2+2\gamma^2+2\beta\gamma+2\gamma\alpha+2\alpha\beta}{4(\alpha+\beta+\gamma)} = \text{etc.}$$

2. Neglecting the atmospheric pressure, the depth of the C.P. (by Art. 162) $= \dfrac{x^2+xy+y^2}{2(x+y)}$ and the thrust there $= w\Delta\dfrac{x+y}{3}$. The atmospheric pressure causes an additional thrust $w\Delta\,.\,h$ at a depth $\dfrac{x+y}{3}$.

$$\therefore \text{ required depth} = \frac{\dfrac{x+y}{3}\cdot\dfrac{x^2+xy+y^2}{2(x+y)} + h\dfrac{x+y}{3}}{\dfrac{x+y}{3}+h} = \text{etc.}$$

3. Let $ABCD$ be the rhombus, A being in the surface and $AC=h$. The depths of the C.P. of Δ^s ABC, ACD are each the same as that of

the rhombus. By Art. 162 the depth of C.P. of $\triangle ACD$ is that of $\|^1$ forces proportional to $\frac{h}{4}, \frac{3h}{4}, \frac{h}{2}$ at the middle points of AD, DC, CA respectively, and thus

$$=\frac{\frac{h}{4}\cdot\frac{h}{4}+\frac{3h}{4}\cdot\frac{3h}{4}+\frac{h}{2}\cdot\frac{h}{2}}{\frac{h}{4}+\frac{3h}{4}+\frac{h}{2}}=\frac{7h}{12}. \quad \therefore \text{ etc.}$$

4. Let $ABCD$ be the square, A being the highest point. If $AC=d$, the depth of $A=d$, that of $C=2d$ and that of $B=\frac{3d}{2}$. The depths of the middle points of AC, CB, BA are $\frac{3d}{2}$, $\frac{7d}{4}$, and $\frac{5d}{4}$. Hence the depth of the C.P. of the square = that of the $\triangle ABC$

$$=\frac{\left(\frac{3d}{2}\right)^2+\left(\frac{7d}{4}\right)^2+\left(\frac{5d}{4}\right)^2}{\frac{3d}{2}+\frac{7d}{4}+\frac{5d}{4}}=d\,\frac{6^2+7^2+5^2}{4(6+7+5)}=\frac{55}{36}d$$

$$=\frac{55}{54}\times\frac{3d}{2}=\frac{55}{54}\times\text{depth of the centre of the square.}$$

5. The depth of the highest point of the rhombus is $h-\frac{a}{2}$, and those of the two successive vertices are h and $h+\frac{a}{2}$. The depths of the middle points of the $\triangle$ thus formed are $h-\frac{a}{4}$, $h+\frac{a}{4}$ and h. Hence the depth of the C.P. required

$$=\frac{\left(h-\frac{a}{4}\right)^2+\left(h+\frac{a}{4}\right)^2+h^2}{\left(h-\frac{a}{4}\right)+\left(h+\frac{a}{4}\right)+h}=h+\frac{a^2}{24h}.$$

6. Let $ABCD$ be the $\|^{\text{gm}}$ so that $h_1+h_3=2h$ and $h_2+h_4=2h$. The C.P. of the $\|^{\text{gm}}$ is the centre of forces at the middle points of AB, AD, BD, BC, CD, DB proportional to

$$\frac{h_1+h_2}{2},\quad \frac{h_1+h_4}{2},\quad h,\quad \frac{h_2+h_3}{2},\quad \frac{h_3+h_4}{2},\quad h,$$

and thus

$$=\frac{\left(\frac{h_1+h_2}{2}\right)^2+\left(\frac{h_1+h_4}{2}\right)^2+h^2+\left(\frac{h_2+h_3}{2}\right)^2+\left(\frac{h_3+h_4}{2}\right)^2+h^2}{\frac{h_1+h_2}{2}+\frac{h_1+h_4}{2}+h+\frac{h_2+h_3}{2}+\frac{h_3+h_4}{2}+h}$$

$$=\frac{1}{4}\,\frac{2h_1^2+2h_2^2+2h_3^2+2h_4^2+8h^2+2h_1(h_2+h_4)+2h_3(h_2+h_4)}{h_1+h_2+h_3+h_4+2h}$$

$$=\frac{1}{4}\,\frac{2h_1^2+2h_2^2+2h_3^2+2h_4^2+8h^2+2(h_1+h_3)\times 2h}{2h+2h+2h}$$

$$=\frac{1}{4}\,\frac{2h_1^2+2h_2^2+2h_3^2+2h_4^2+8h^2+8h^2}{6h}=\text{etc.}$$

7. Let $ABCD$ be the half-hexagon, BC being the side in the surface. Let $2a\left[=\frac{a\sqrt{3}}{2}\right]$ be the depth of O, the middle point of DA; join OB and OC.

For the three triangles OAB, OBC, OCD on placing wts. proportional to the depths at the middle points of the sides we have $6a$ at depth a and $4a$ at depth $2a$.

$$\therefore \text{ required depth}=\frac{6a\,.\,a+4a\,.\,2a}{6a+4a}=\frac{14}{10}a=\frac{7}{5}\cdot\frac{a\sqrt{3}}{4}=\frac{7\sqrt{3}}{20}a.$$

8. Let $ABCD$ be the rhombus, A being the highest point; let $2a$ be its area and d the length AC. If ρ and 3ρ be the densities of the upper and lower liquids, we may consider the rhombus as immersed in two liquids, one of density ρ extending all over the rhombus, and one of density 2ρ over the lower half only. The thrust due to the first is proportional to $2a\,.\,\rho\,.\,\frac{d}{2}$, *i.e.* $a\rho d$ and acts, by Ex. 3, at a depth $\frac{7d}{12}$.

The thrust due to the latter is $a\,.\,2\rho\,.\,\frac{d}{6}$ and acts at a depth [by Art. 153] $\frac{d}{4}$ below BD, and therefore at a depth $\frac{3d}{4}$ below A.

Therefore depth of required C.P. below A

$$=\frac{a\rho d\,.\,\frac{7d}{12}+\frac{a\rho d}{3}\times\frac{3d}{4}}{a\rho d+\frac{a\rho d}{3}}=\frac{3}{4}\left[\frac{7}{12}+\frac{1}{4}\right]d=\frac{5}{8}d.\quad\therefore\text{ etc.}$$

10. For the first part see solution to Ex. 11 with $\alpha=\beta=0$.

If the second part be possible, we have

$$3h=2\left[\frac{\gamma+\delta}{2}-\frac{1}{6}\frac{\gamma\delta}{n}\right].$$

$$\therefore\ 9h^2-3\gamma-3\delta+\gamma\delta=0.$$

Therefore $h=\frac{\gamma}{3}$ or $\frac{\delta}{3}$, *i.e.* depth of the C.G. of $ABCD$ is the same as either that of the $\triangle ACB$ or of $\triangle ABD$, either of which is clearly impossible.

12. Let $ABCD$ be the square, A being the highest vertex. Let O be the middle point of AC, and let θ be the inclination of AC to the vertical or of BD to the horizontal. Let E, F, G, H be the middle points of AB, BC, CD, DA.

The depths of A, B, C, D are

$$h-a\sqrt{2}\cos\theta,\quad h+a\sqrt{2}\sin\theta,\quad h+a\sqrt{2}\cos\theta,\quad h-a\sqrt{2}\sin\theta,$$

and thus those of E, F, G, H are

$$h-a\sqrt{2}\,\frac{\cos\theta-\sin\theta}{2},\quad h+a\sqrt{2}\,\frac{\cos\theta+\sin\theta}{2},\quad h+a\sqrt{2}\,\frac{\cos\theta-\sin\theta}{2},$$

and $$h-a\sqrt{2}\,\frac{\cos\theta+\sin\theta}{2}.$$

Thus sum of depths of E and G = sum of depths of F and $H=2h$ and sum of squares of depths of E and G = sum of squares of depths of F and $H=2h^2+a^2$.

Now since the $\triangle$s ABC, ACD are equal in area we have to place particles proportional to the depths at the middle points of the sides, and thus depth of C.P.

$$=\frac{\text{sum sqs. of depths of } E,\ F,\ G,\ H+2\text{ sq. depth of } O}{\text{sum of depths of } E,\ F,\ G,\ H+2\text{ depth of } O}$$

$$=\frac{4h^2+2a^2+2h^2}{4h+2h}=h+\frac{a^2}{3h}.$$

Again the horizontal distances of A, B, C, D from O are

$$-a\sqrt{2}\sin\theta,\quad -a\sqrt{2}\cos\theta,\quad a\sqrt{2}\sin\theta\ \text{ and }\ a\sqrt{2}\cos\theta.$$

Hence the horizontal distances of E, F, G, H are

$$-\frac{a}{\sqrt{2}}(\cos\theta+\sin\theta),\quad -\frac{a}{\sqrt{2}}(\cos\theta-\sin\theta),\quad \frac{a}{\sqrt{2}}(\cos\theta+\sin\theta)$$

and $$\frac{a}{\sqrt{2}}(\cos\theta-\sin\theta).$$

Hence it easily follows that the sum of the products of the masses at E, F, G, H into their horizontal distances from O is zero. Hence the horizontal distance of the C.P. is zero, *i.e.* the C.P. is vertically under O.

13. Let $ABCD$ be the square, O its centre, A being the highest, and C the lowest vertex, and B being also immersed. Let $2a$ be the side of the square and θ the inclination to the horizontal of the line KL, where K, L are the middle points of CD and AB. Let the water-line through O cut AB in Q and CD in P. Let Δ_1, Δ_2, Δ_3 be the area of the Δ^s OQB, OBC and OCP.

Then $$OQ = OP = \frac{a}{\cos\theta}, \quad \angle QOB = 45^\circ - \theta,$$

and $$\angle POC = 45^\circ + \theta.$$

Hence $$\frac{\Delta_1}{\cos\theta - \sin\theta} = \frac{\Delta_2}{2\cos\theta} = \frac{\Delta_3}{\cos\theta + \sin\theta}.$$

Let E, F, G, H, M, N, S be the middle points of OQ, QB, BO, BC, CO, CP, PO respectively. Then the depths of F, G, H, M, N are respectively

$$\frac{a}{2}(\cos\theta - \sin\theta), \quad \frac{a}{2}(\cos\theta - \sin\theta), \quad a\cos\theta, \quad \frac{a}{2}(\cos\theta + \sin\theta),$$

and $$\frac{a}{2}(\cos\theta + \sin\theta).$$

Hence, by Art. 162, we have at each of F and G a force

$$\frac{1}{3}\Delta_1 \cdot \frac{a}{2}(\cos\theta - \sin\theta), \text{ which } \propto (\cos\theta - \sin\theta)^2,$$

at G a force $\frac{1}{3}\Delta_2 \cdot \frac{a}{2}(\cos\theta - \sin\theta)$, which $\propto 2\cos\theta(\cos\theta - \sin\theta)$,

at H a force $\frac{1}{3} \cdot \Delta_2 \cdot a\cos\theta$, which $\propto 4\cos^2\theta$,

at M a force $\frac{1}{3}\Delta_2 \times \frac{a}{2}(\cos\theta + \sin\theta)$, which $\propto 2\cos\theta(\cos\theta + \sin\theta)$,

and at each of M and N a force

$$\frac{1}{3}\Delta_3 \times \frac{a}{2}(\cos\theta + \sin\theta), \text{ which } \propto (\cos\theta + \sin\theta)^2.$$

Also the horizontal distances from O of F, G, H, M, N are

$$-\frac{a}{2}\left(\frac{1}{\cos\theta} + \cos\theta + \sin\theta\right), \quad -\frac{a}{2}(\cos\theta + \sin\theta), \quad -a\sin\theta,$$

$$\frac{a}{2}(\cos\theta - \sin\theta), \quad \text{and} \quad \frac{a}{2}\left(\frac{1}{\cos\theta} + \cos\theta - \sin\theta\right).$$

Hence the sum of the products of each force into the horizontal distance of its point of application varies as

$$(\cos\theta-\sin\theta)^2\left[-\frac{a}{2\cos\theta}-a(\cos\theta+\sin\theta)\right]$$
$$+\cos\theta(\cos\theta-\sin\theta)[-a\cos\theta-a\sin\theta]+4\cos^2\theta(-a\sin\theta)$$
$$+\cos\theta(\cos\theta+\sin\theta)(a\cos\theta-a\sin\theta)$$
$$+(\cos\theta+\sin\theta)^2\left[\frac{a}{2\cos\theta}+a(\cos\theta-\sin\theta)\right],$$

and this expression reduces to zero.

The horizontal distance of the centre of pressure is thus zero and hence the centre of pressure is always vertically beneath O.

EXAMPLES. XXXII. (Pages 222—227.)

1. From Art. 167, Ex. 1, we have

$$\pi\rho r^2\left[gh+\frac{\omega^2r^2}{4}\right]=\frac{3}{2}\times\pi\rho r^2[gh+0].$$
$$\therefore \frac{\omega^2r^2}{4}=\frac{1}{2}gh,\quad i.e.\ \omega=\sqrt{\frac{2gh}{r^2}}.$$

2. Here, from the same example, we have

$$\pi\rho r^2\left[gh+\frac{\omega^2r^2}{4}\right]=5\times\frac{1}{4}\pi\rho r^4\omega^2.$$
$$\therefore\ gh+\frac{\omega^2r^2}{4}=\frac{5}{4}\times r^2\omega^2,$$
$$i.e.\ \omega=\frac{\sqrt{gh}}{r}.$$

3. Here
$$gh+\frac{\omega^2r^2}{4}=\frac{n}{4}\times r^2\omega^2.$$
$$\therefore\ \omega=\frac{2}{r}\sqrt{\frac{gh}{n-1}}.$$

4. With the fig. of Case I. of Ex. 2 of Page 218 we have (by Art. 166) pressure at a pt. of the base distant y from O

$$=\rho\left[\frac{1}{2}\omega^2\cdot y^2+g\cdot AO\right].$$

Also
$$AO=h-AN=h-\frac{\omega^2r^2}{2g}.$$
$$\therefore\ \text{press.}=\rho\left[\frac{1}{2}\omega^2y^2+gh-\frac{\omega^2r^2}{2}\right]=\text{etc.}$$

5. Let a section through the vertex V of the cone cut the base in the line ACB, C being the centre. Through V draw a parabola, with axis vert. and lat.-rect. $\frac{2g}{\omega^2}$, and let vertical lines through A, B meet it in K, L. Let KL meet axis in N.

Then $$a^2 = NL^2 = \frac{2g}{\omega^2} \cdot NV, \quad i.e.\ VN = \frac{\omega^2 a^2}{2g}.$$

Then press. on base = wt. of the volume $ABLVK$ = wt. of (cylinder BK − paraboloid KVL).

$$\therefore\ 6 \,.\, \text{vol. } AVB = \text{cylinder } BK - \text{paraboloid } KVL.$$

$$\therefore\ 6\pi a^2 \cdot \frac{1}{3} a \cot\alpha = \pi a^2 [a \cot\alpha + VN] - \frac{1}{2}\pi a^2 \,.\, VN.$$

$$\therefore\ a \cot\alpha = \frac{1}{2} \,.\, VN = \frac{\omega^2 a^2}{4g}.$$

$$\therefore\ \omega = \sqrt{\frac{4g \cot\alpha}{a}}.$$

6. In Ex. 2, Page 218, the vertex A coincides, as in Case 2, with O when $\frac{\omega^2 r^2}{2g} = h$, *i.e.* when $\omega = \frac{\sqrt{2gh}}{r}$; and then the volume of the paraboloid PAP' just $= \frac{1}{2}$ that of the cylinder. Hence in this case the water left in the cylinder is of volume half that of the cylinder.

7. With the figure of Ex. 1, Art. 167, the lid will turn round the hinge B if its weight = the weight of water of volume $P'APEAB$

$$= \frac{1}{2}\pi a^2 \,.\, AN \,.\, g\rho = \frac{\pi a^2 g\rho}{2} \times \frac{\omega^2 a^2}{2g} = \frac{1}{4}\pi a^4 \omega^2 \rho.$$

8. As in Ex. 6, we have $\Omega = \frac{\sqrt{2gh}}{r}$.

By Page 218, Ex. 2, Case 3, if volume left in

$$= \frac{1}{n}\text{th original vol.}$$

$$= \frac{1}{n} \times \frac{\pi}{2} r^2 h,$$

then $$\frac{\pi}{2n} r^2 h = \pi r^2 h - \text{vol. } PQQ'P'$$

$$= \pi r^2 h - \frac{\pi h}{\omega^2}[\omega^2 r^2 - gh]$$

$$= \frac{\pi g h^2}{\omega^2};$$

and therefore $$\omega = \frac{\sqrt{2ngh}}{r} = \sqrt{n} \,.\, \Omega.$$

9. Let the tube, of centre O, revolve round the tangent at C. Let the parabolic free surface cut tangent at C in A and the circle in P, Q, so that the two latter points are at the ends of a diameter. Let $\theta = \angle PQ$ makes with CO produced. Draw PN, $QM \perp^r$ to CA, so that $PN = a + a\cos\theta$, $QM = a - a\cos\theta$.

$$\therefore\ (a + a\cos\theta)^2 = \frac{2g}{\omega^2} \cdot AN \text{ and } (a - a\cos\theta)^2 = \frac{2g}{\omega^2} \cdot AM.$$

$$\therefore, \text{ by subtraction, } 4a^2\cos\theta = \frac{2g}{\omega^2}[AM - AN]$$

$$= \frac{2g}{\omega^2} \cdot MN = \frac{2g}{\omega^2} \cdot 2a\sin\theta. \quad \therefore \text{ etc.}$$

10. Let the parabolic free surface meet the tube in P and Q, and the vertical through the centre C in A. Draw PN, $QM \perp^r$ to CA. The $\angle PCN = 60°$ and $\angle QCM = 30°$, so that $PN = \dfrac{a \cdot \sqrt{3}}{2}$ and $QM = \dfrac{a}{2}$.

$$\therefore\ \frac{3a^2}{4} = PN^2 = \frac{2g}{\omega^2} \cdot AN,$$

and
$$\frac{a^2}{4} = QM^2 = \frac{2g}{\omega^2} \cdot AM.$$

$$\therefore\ \frac{3a^2}{4} - \frac{a^2}{4} = \frac{2g}{\omega^2} \cdot MN = \frac{2g}{\omega^2}\left[\frac{a\sqrt{3}}{2} + \frac{a}{2}\right]. \quad \therefore \text{ etc.}$$

11. Let C be the centre of the tube, A its lowest point. The parabolic free surface will have its vertex at A and pass through a point P on the tube such that $\angle ACP = \dfrac{\theta}{2}$. Draw $PN \perp^r$ to AC. Then

$$a^2\sin^2\frac{\theta}{2} = PN^2 = \frac{2g}{\omega^2} \cdot AN = \frac{2g}{\omega^2}\left(a - a\cos\frac{\theta}{2}\right)$$

$$= \frac{2g}{\omega^2} \cdot 2a\sin^2\frac{\theta}{4}.$$

$$\therefore\ \omega^2 = \frac{g}{a} \frac{4\sin^2\dfrac{\theta}{4}}{\sin^2\dfrac{\theta}{2}} = \frac{g}{a} \cdot \frac{1}{\cos^2\dfrac{\theta}{4}}. \quad \therefore \text{ etc.}$$

12. Let CB and CP be the vertical and horizontal radii of the tube. Draw a parabola, lat.-rect. $\dfrac{2g}{\omega^2}$, to go through P and cut the tube in Q and CB in A. Draw $QN \perp^r$ to CA. Then, by the question, $\angle BCQ = 45°$. $\therefore\ QN = \dfrac{a}{\sqrt{2}}$. Hence

$$a^2 = CP^2 = \frac{2g}{\omega^2} \cdot AC, \text{ and } \frac{a^2}{2} = \frac{2g}{\omega^2} \cdot AN.$$

$$\therefore \frac{a^2}{2}=\frac{2g}{\omega^2}.CN=\frac{2g}{\omega^2}.\frac{a}{\sqrt{2}}. \quad \therefore \omega^2=\frac{2g}{a}.\sqrt{2}.$$

Also $$CA=\frac{\omega^2 a^2}{2g}=a\sqrt{2}.$$

13. With the notation of Ex. 12, here $\angle QCN=60°$.

$$\therefore a^2=CP^2=\frac{2g}{\omega^2}.AC, \text{ and } \frac{a^2.3}{4}=QN^2=\frac{2g}{\omega^2}.AN.$$

$$\therefore \frac{a^2}{4}=\frac{2g}{\omega^2}.CN=\frac{2g}{\omega^2}.a.\frac{1}{2}.$$

$$\therefore \omega=2\sqrt{\frac{g}{a}}.$$

14. Let BC, CD, DE be the sides of the tube; O the middle pt. of the horizontal side CD. Through E, B draw the free parabolic surface to meet CD in P and Q and the vertical through O in A. Then, if EB meet this vertical in N,

$$\left(\frac{a}{2}\right)^2=EN^2=\frac{2g}{\omega^2}.AN.$$

$$\therefore AO=\frac{\omega^2 a^2}{8g}-a,$$

and $$OP^2=\frac{2g}{\omega^2}.AO=\frac{a^2}{4}-\frac{2ga}{\omega^2}.$$

$$\therefore \text{length that escapes}=2.OP=a\sqrt{1-\frac{8g}{\omega^2 a}}.$$

This value is imaginary, *i.e.* nothing escapes, if $\omega>\sqrt{\frac{8g}{a}}$.

15. With the fig. of Ex. 4, through E draw the free surface, of lat. rect. $\frac{2g}{\omega^2}$ and axis BC, to meet CB in A. Then $a^2=EB^2=\frac{2g}{\omega^2}.AB$.

Hence length that flows out $=\frac{\omega^2 a^2}{2g}$.

If however $\frac{\omega^2 a^2}{2g}>BC$, *i.e.* $\omega^2>\frac{2g}{a}$, then A falls below C and the parabola meets CD in P, where

$$CP^2=\frac{2g}{\omega^2}.CA=\frac{2g}{\omega^2}AB-\frac{2g}{\omega^2}.a=a^2-\frac{2ga}{\omega^2}.$$

Thus length that flows out in this case $=BC+CP=$ etc.

16. Let a vertical plane through the axis cut the cylinder in the lines PC, CB, BP' and let O be the middle point of BC; and let PP' meet the axis ON in N.

Let the free surface through P, P' meet axis in A. If $b > \frac{1}{2}h$, *i.e.* if the quantity of the liquid > half the cylinder, then A is above O, and

$$\pi a^2 (h-b) = \text{vol. } PAP' = \pi a^2 . \frac{1}{2} AN.$$

$$\therefore\ h - b = \frac{1}{2} . AN = \frac{1}{2} . \frac{\omega^2}{2g} . PN^2 = \frac{\omega^2 a^2}{4g}.$$

$$\therefore\ \omega = \frac{2}{a}\sqrt{g(h-b)}.$$

If $b < \frac{1}{2}h$, A is below O as in the second figure of Page 218, and then

$$\pi a^2 (h-b) = \text{vol. } PQQ'P$$

$$= \frac{\pi h}{\omega^2}(\omega^2 a^2 - gh), \text{ as on Page 219.}$$

$$\therefore\ \omega = \frac{h}{a}\sqrt{\frac{g}{b}}.$$

17. Here $\left(\frac{a}{2}\right)^2 = \frac{2g}{\omega^2} . a$, so that $\omega = \sqrt{\frac{8g}{a}}$.

18. Let a vert. section of the cone through the vertex V cut the base in PP', the axis of the cone meeting the parabolic free surface which in the limiting case goes through P, P' in A and PP' in N. Then

$$h^2 \tan^2 \alpha = PN^2 = \frac{2g}{\omega^2} . AN.$$

Also it is given that paraboloid PAP' = half cone,

i.e. $$\pi a^2 . \frac{1}{2} AN = \frac{1}{2} . \pi a^2 . \frac{1}{3} VN.$$

$$\therefore\ \frac{\omega^2 h^2 \tan^2 \alpha}{2g} = \frac{1}{3} . h.$$

$$\therefore\ \omega^2 = \frac{2}{3}\frac{g}{h}\cot^2 \alpha.$$

If $\omega > \sqrt{\frac{2g}{3h}} \cot \alpha$, the water overflows.

19. Let the plane of the paper cut the base of the hemisphere in the points P and P'. Through P and P' draw a parabola, of latus-rectum $\frac{2g}{\omega^2}$, and let it meet the axis through the centre C of the hemisphere in A. Then

$$a^2 = CP^2 = \frac{2g}{\omega^2} \cdot CA, \quad i.e.\ CA = \frac{\omega^2 a^2}{2g}.$$

$\therefore$ amount of liquid that has run over

$$= \text{vol. of paraboloid}\ . PAP' = \pi a^2 \times \tfrac{1}{2} CA$$

$$= \frac{1}{4} \frac{\pi \omega^2 a^4}{g}.$$

20. With the figure of Ex. 18, we have

$$h^2 \tan^2 \alpha = PN^2 = \frac{2g}{\omega^2} \cdot AN.$$

$\therefore$ amount that flows over = paraboloid PAP'

$$= \pi h^2 \tan^2 \alpha \times \frac{1}{2} AN = \frac{1}{4} \frac{\pi \omega^2 h^4 \tan^4 \alpha}{g}.$$

This holds provided that the parabola does not meet the line VP again in a point between V and P. In the limiting case VP is a tangent to the parabola so that then $VA = AN$, by a property of all parabòlas,

i.e.
$$h = 2AN = \frac{\omega^2 h^2 \tan^2 \alpha}{g},$$

and
$$\therefore \omega = \sqrt{\frac{g}{h}} \cdot \cot \alpha.$$

21. Let the cup be formed by the revolution of the parabola PAP' about its axis AN. Draw $PN \perp^r$ to the axis and through P, P' draw the free surface, *viz.* a parabola of latus-rectum $\frac{2g}{\omega^2}$. Let vertex of the latter be a point A' on AN.

Then we have

$$4ah = PN^2 = \frac{2g}{\omega^2} \cdot A'N \quad \text{.........................} \quad (1).$$

Also, since none of the liquid is spilt,

$\therefore$ paraboloid PAP' − paraboloid $PA'P'$ = total quantity of liquid,

i.e.
$$\pi \cdot 4ah \cdot \frac{1}{2} h - \pi \cdot 4ah \cdot \frac{1}{2} A'N$$

$$= \pi \cdot 4a \frac{h}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} h,$$

i.e. $$A'N = \frac{3h}{4}.$$

$$\therefore \text{ (1) gives } 4a = \frac{2g}{\omega^2} \cdot \frac{3}{4}, \; i.e. \; \omega = \frac{1}{4}\sqrt{\frac{6g}{a}}.$$

22. Let the cup be formed by the revolution of the parabolic arc PCQ about its axis CN. When it and the water are revolving, through P and Q draw the parabolic free surface, latus-rectum $\frac{2g}{\omega^2}$, with its vertex A on CN. If A lies above C, there is always water at C until the whole amount has gone out through a hole at C. If A be above C, and λ be the latus-rectum of the parabola of the cup, then

$$\lambda . NC = PN^2 = \frac{2g}{\omega^2} . AN.$$

$$\therefore \lambda = \frac{2g}{\omega^2} \cdot \frac{AN}{NC}, \; i.e. \; \lambda < \frac{2g}{\omega^2}.$$

23. Let the bowl be the figure formed by the rotation of the semi-circle ADB about its axis CD, C being the centre. Since the bowl is only just full of water, the pressure will be just zero where it is least, *viz.* at the highest point D. Through D as vertex draw a parabola, with its axis vertical and latus-rectum $\frac{2g}{\omega^2}$, and let vertical lines through A, B meet it in K and L. Since the bowl is on the point of rising, its weight = upward pressure on bowl

= wt. of $AKDLBDA$

= wt. of (cylinder $ABLK - \frac{1}{2}$ sphere ADB − paraboloid KDL)

$= w \,.\, \pi a^2 \left[AK - \frac{2}{3}a - \frac{1}{2} . DN\right]$, where KL meets CD produced in N.

Now $$a^2 = \frac{2g}{\omega^2} . ND.$$

$$\therefore \frac{\text{wt. of hemisphere}}{\text{wt. contained water}} = \frac{AK - \frac{2}{3}a - \frac{1}{2}DN}{\frac{2}{3}a} = \frac{\frac{a}{3} + \frac{1}{2}DN}{\frac{2}{3}a}$$

$$= \frac{1}{2} + \frac{3}{4} \cdot \frac{\omega^2 a}{2g} = \frac{4g + 3\omega^2 a}{8g}.$$

24. Let the cup be the figure formed by the revolution of the parabola PAP' about its axis AN. Through A draw the free surface, a parabola with the vertical through A as axis, A as vertex, and $\frac{2g}{\omega^2}$ as

latus-rectum. Through P, P' draw vertical lines PK, $P'L$ to meet this parabola in K, L, and let KL meet NA produced in M. Then the cup will be on the point of rising when W = upward press. of liquid

$$= \text{wt. of [cylinder } PKLP' - \text{paraboloid } PAP' - \text{paraboloid } KAL]$$

$$= w_1 . \pi . PN^2 \left[NM - \frac{1}{2} AN - \frac{1}{2} AM\right],$$

where w_1 = wt. of unit vol. of liquid,

$$= w_1 . \pi . PN^2 . \frac{1}{2} [AN + AM] \quad \text{.................... (1).}$$

Let $AN = h$, so that $PN^2 = 4ah$, and

$$AM = \frac{\omega^2}{2g} . MK^2 = \frac{\omega^2}{2g} . 4ah,$$

and $$\therefore W = w_1 . \pi . 2ah . \left[h + \frac{\omega^2}{g} . 2ah\right].$$

Also $$w = w_1 . \pi . 4ah . \frac{1}{2} h.$$

$$\therefore \frac{W}{w} = 1 + \frac{\omega^2}{g} . 2a, \quad i.e. \quad \omega = \sqrt{\frac{W - w}{w} . \frac{g}{2a}}.$$

26. Let the section of the cone by a plane through its axis VAN be VP and VP'. Through P, P' draw a parabola of latus-rect. $\frac{2g}{\omega^2}$ having its vertex A on VN.

Then $$\frac{h^2}{3} = h^2 \tan^2 30° = PN^2 = \frac{2g}{\omega^2} . AN,$$

so that $$AN = \frac{\omega^2 h^2}{6g} \quad \text{.................... (1).}$$

For equilibrium we then have

$$\rho \times \frac{1}{3} \pi . PN^2 . h = \text{upward pressure of the liquid}$$

$$= \frac{4\rho}{3} [\text{vol. cone} - \text{vol. } PAP'].$$

$$\therefore \frac{1}{3} h = \frac{4}{3} \left[\frac{1}{3} h - \frac{1}{2} AN\right].$$

$$\therefore \frac{h}{3} = 2 . AN = \frac{\omega^2 h^2}{3g}.$$

$$\therefore \omega = \sqrt{\frac{g}{h}}.$$

28. Let the string OP joining the small sphere P to the fixed point O make an angle θ with the vertical through O drawn downwards. Then, as in the last example, we have

$$mg - T\cos\theta = \frac{mg}{\sigma} \quad \text{.......................} \quad (1),$$

and
$$m\omega^2 y = \frac{m}{\sigma}\omega^2 y + T\sin\theta,$$

i.e.
$$m\omega^2 l = \frac{m\omega^2 l}{\sigma} + T \quad \text{.......................} \quad (2).$$

(1) and (2) give

$$mg\left(1 - \frac{1}{\sigma}\right) = \cos\theta \,.\, m\omega^2 l\left(1 - \frac{1}{\sigma}\right).$$

$$\therefore \cos\theta = \frac{g}{\omega^2 l} \quad \text{.............................} \quad (3).$$

This always gives a real value for θ if

$$\omega^2 l > g, \text{ i.e. if } \omega > \sqrt{\frac{g}{l}}.$$

Let the sphere be slightly displaced upwards, so that θ is increased and $\cos\theta$ decreased. Then, since by (2) T is constant, the vertical pull $T\cos\theta$ of the tension is lessened, the left hand of (1) is thus increased, and the downward force is increased and the sphere returns towards the position given by (3). Similarly if θ is decreased, ∴ etc.

EXAMPLES. XXXIII. (Page 234.)

1. If x be the depth immersed, then

$$x \,.\, \sigma = \rho \,.\, 2b.$$

$$HM = \frac{\lambda \,.\, 2a \,.\, \frac{a^2}{3}}{\lambda \,.\, x \,.\, 2a}, \text{ where } \lambda \text{ is the thickness, } = \frac{a^2}{6b} \cdot \frac{\sigma}{\rho}.$$

The equilibrium is thus stable if

$$HM > b - \frac{x}{2},$$

i.e. if
$$\frac{a^2}{6b} \cdot \frac{\sigma}{\rho} > b - \frac{\rho}{\sigma} b,$$

i.e. if
$$\frac{a^2}{6b^2} > \frac{\rho}{\sigma} - \frac{\rho^2}{\sigma^2}.$$

2. The depth immersed $=\dfrac{c}{2}$.

Hence
$$HM=\frac{ab\,.\,\dfrac{b^2}{12}}{ab\,.\,\dfrac{c}{2}}=\frac{1}{6}\cdot\frac{b^2}{c},$$

for a rotation round an axis parallel to the side a.

The equilibrium is thus stable if
$$HM>\frac{c}{2}-\frac{c}{4}>\frac{c}{4},$$

i.e. if
$$b^2>\frac{3}{2}c^2,$$

i.e. if
$$b>\frac{1}{4}\sqrt{6}\,.\,c.$$

The condition of stability for a rotation about the axis parallel to b, viz. $a>\dfrac{1}{4}\sqrt{6}\,.\,c$, is then clearly satisfied, since $a>b$.

3. If $2l$ be the length, and a the radius of the base, of the cylinder, then
$$HM=\frac{4al\,.\,\dfrac{l^2}{3}}{\dfrac{1}{2}\pi a^2\,.\,2l}=\frac{4}{3\pi}\cdot\frac{l^2}{a}.$$

Also it can be shewn that the distance of the C. G. of a semi-circle from the centre is $\dfrac{4a}{3\pi}$.

Therefore equilibrium is stable if
$$\frac{l^2}{a}>a,\ \text{if } l>a,$$

i.e. if height $>$ diameter of base.

4. x being the length of the axis immersed, we have
$$\frac{b}{h-x}=\frac{a}{h}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (1),$$

and
$$\left\{\frac{1}{3}\pi a^2h-\frac{1}{3}\pi b^2(h-x)\right\}\sigma=\frac{1}{3}\pi a^2h\rho,$$

so that
$$\{h^3-(h-x)^3\}\,\sigma=h^3\rho\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Also
$$HM=\frac{\pi b^2\dfrac{b^2}{4}}{\dfrac{1}{3}\pi a^2h-\dfrac{1}{3}\pi b^2(h-x)}=\frac{3}{4}\,\frac{b^4}{a^2h-b^2(h-x)}.$$

Also if V be the vertex

$$\frac{1}{3}\pi b^2(h-x)\cdot\frac{3}{4}(h-x)+\left[\frac{1}{3}\pi a^2h-\frac{1}{3}\pi b^2(h-x)\right]VH=\frac{1}{3}\pi a^2h\cdot\frac{3}{4}h.$$

$$\therefore\quad VH=\frac{3}{4}\times\frac{a^2h^2-b^2(h-x)^2}{a^2h-b^2(h-x)}.$$

$$\therefore\quad HG=\frac{3}{4}\frac{a^2h^2-b^2(h-x)^2}{a^2h-b^2(h-x)}-\frac{3}{4}h=\frac{3}{4}\frac{xb^2(h-x)}{a^2h-b^2(h-x)}.$$

Therefore equilibrium is stable if $HM>HG$,

i.e. if $\qquad b^2>x(h-x),$

i.e. if $\qquad a^2(h-x)>h^2x,$ by (1),

i.e. if $\qquad a^2h>(h^2+a^2)x,$

i.e. if $\qquad x<h\sin^2\alpha,$

and therefore $\qquad h-x>h\cos^2\alpha;$

and then (2) gives

$$\frac{\rho}{\sigma}<\frac{1}{h^3}[h^3-h^3\cos^6\alpha],$$

i.e. $$\frac{\rho}{\sigma}<1-\cos^6\alpha.$$

5. The movement produces the same result as the placing of $-m$ tons on one side of the ship and of $+m$ tons on the other. The moment of the thus altered weight of the ship about G

$$=\text{moment of this couple}=mlg.$$

This is balanced by the moment of the buoyancy acting at the new centre of buoyancy H', and this moment

$$=Mg\times GM\sin M'MG.$$

$$\therefore\quad ml=Mh\sin M'MG.$$

$$\therefore\quad \angle M'MG=\sin M'MG=\frac{ml}{Mh},$$

since the $\angle M'MG$ is small.

6. Here $M=9000$, $m=20$, $l=42$.

Also $$\frac{10}{12}=20\times\angle M'MG.$$

Therefore from the result of the previous question

$$\frac{10}{240}=\frac{20}{9000}\cdot\frac{42}{h}.$$

$$\therefore\quad h=\frac{24\times20\times42}{9000}=\frac{16\times14}{100}=2\cdot24\text{ ft.}$$

EXAMPLES. XXXIV. (Page 239.)

1. Let r, k be the radius and thickness of the second, and $3r$, $2k$ those of the first. If t be the common tensile strength, then

$$p_1=\frac{2 \cdot t \cdot 2k}{3r} \text{ and } p_2=\frac{2 \cdot t \cdot k}{r}.$$

$$\therefore \frac{p_2}{p_1}=\frac{3}{2}.$$

2. $t=12000$, $r=6$.

$$\therefore p=\frac{t \cdot k}{r}=\frac{12000\times\frac{1}{4}}{6}=500 \text{ lbs. per sq. in.}$$

3.
$$p=200\times 62\tfrac{1}{2} \text{ lbs. per sq. ft.}$$

$$=200\times 62\tfrac{1}{2}\times\frac{1}{144} \text{ lbs. per sq. in.}$$

$$\therefore 200\times 62\tfrac{1}{2}\times\frac{1}{144}=\frac{10000\times k}{4},$$

where k is the thickness in inches.

$$\therefore k=\frac{100\times 125\times 4}{10000\times 144}=\frac{5}{144} \text{ in.}$$

4. Here
$$p_0=\frac{2t_0}{a}, \text{ and } p_1=\frac{2t_1}{r}.$$

$$\therefore \frac{p_1}{p_0}=\frac{a}{r}\cdot\frac{t_1}{t_0}=\frac{a}{r}\cdot\frac{r^2}{a^2}=\frac{r}{a}.$$

Also, by Art. 114,

$$p_0 \cdot a^3=\frac{p_1 r^3}{1+at}.$$

$$\therefore 1+at=\frac{p_1}{p_0}\cdot\frac{r^3}{a^3}=\frac{r^4}{a^4}. \quad \therefore \text{ etc.}$$

MISCELLANEOUS EXAMPLES. (Pages 240—248.)

1. If x be the depth, then, by Art. 57,

$$x\times 1{\cdot}025=100\times{\cdot}92.$$

$$\therefore x=89\tfrac{31}{41} \text{ yds.}$$

2. Let V be the volume, and ρ the density in grammes per cubic cm. Then

$$V \cdot \rho=1000, \text{ and } \rho \cdot 1=1 \cdot \frac{3}{5}. \quad \therefore \text{ etc.}$$

3. If n be the required fraction, then

$$n \times \cdot 84 = 1 \times \frac{2}{3}. \quad \therefore \; n = \frac{50}{63}.$$

4. Its weight must = thrust due to a pressure 772 – 730, *i.e.* 42 mm. of mercury on its face

$$= \pi \cdot \left(\frac{5}{2}\right)^2 \cdot \frac{42}{10} \text{ cub. cm. wt.}$$

$$= \pi \cdot \frac{25}{4} \cdot \frac{42}{10} \times 13{\cdot}6 \text{ grammes}$$

$$= 357\pi \text{ grammes.}$$

5. Let V and V' be the volumes of gold and silver in the crown, ρ and ρ' the spec. gravities of gold and silver. Then

$$V \cdot 1 + V' \cdot 1 = \frac{1}{14}[V \cdot \rho + V'\rho'] \quad \text{.................(1)},$$

$$V \cdot 1 = \frac{4}{77} \cdot V \cdot \rho \quad \text{.........................(2)},$$

and

$$V' \cdot 1 = \frac{2}{21} \cdot V' \cdot \rho' \quad \text{.......................(3)}.$$

(2) and (3) give

$$\rho = \frac{77}{4}, \quad \rho' = \frac{21}{2},$$

and then (1) gives

$$\frac{V}{V'} = \frac{14 - \rho'}{\rho - 14} = \frac{14 - \frac{21}{2}}{\frac{77}{4} - 14} = \frac{2}{3}.$$

6. Let x inches = external side of cube. If the vessel *just* floats, then

$$[x^3 - (x - 2)^3] \times 2\tfrac{34}{91} = x^3 \times 1, \text{ by Art. 57.}$$

$$\therefore \; 125x^3 = 216\,(x - 2)^3.$$

$$\therefore \; 5x = 6\,(x - 2).$$

$$\therefore \; x = 12 \text{ and } (x - 2)^3 = 10^3 = 1000.$$

7. Since it floats in the first liquid, the density of the body is ρ_1. Hence, by Art. 57,

$$\left[\frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 \cdot 2r\right] \times \rho_1 = \frac{2}{3}\pi r^3 \times \rho_2.$$

$$\therefore \; \rho_2 = 2\rho_1.$$

8. Let A be the section, h the height in cms., and σ the density of the cylinder in grammes per cm. Then

$$A\,.\,h\,.\,\sigma=2=A\,(h-7)\,.\,1,\quad A\,.\,h\,.\,\sigma+\frac{1}{2}=Ah\,.\,1.$$

$$\therefore\ A=\frac{1}{14},\ h=35,\ \text{and}\ \sigma=\frac{4}{5}.$$

If x be the required length, then

$$A\,(h-x)\,.\,5=A\,.\,h\,.\,\sigma.$$

$$\therefore\ h-x=\frac{1}{5}\,.\,35\,.\,\frac{4}{5}=5{\cdot}6,$$

and $$\therefore\ x=35-5{\cdot}6=29{\cdot}4.$$

9. Let V be the required no. of cubic feet. Then upward thrust
$=$wt. of V cubic feet of air $-$ wt. of V cubic feet of the gas
$=(V\times 1\tfrac{1}{4}-V\times 1\tfrac{1}{4}\times{\cdot}52)$ ozs. wt.
$=V\times\frac{5}{4}\times\frac{{\cdot}48}{16}$ lbs. wt. $=\frac{3V}{80}$ lbs. wt.

Hence $$\frac{3V}{80}=1200,$$

so that $$V=32000.$$

10. Let $W=$required wt. in ozs. Then

$$W+8=\text{wt. of }\pi\times\left(\frac{3}{2}\right)^2\times\frac{9}{2}\text{ cub. ins. of water}$$

$$=\pi\times\frac{9}{4}\times\frac{9}{2}\times\frac{1000}{1728}.$$

$$\therefore\ W=\frac{375\pi}{64}-8=\text{etc.}$$

11. Let ρ be the sp. gr. of the wood and x ins. the length immersed in the water. Then

$$4\times\rho+8\times{\cdot}24=x\times 1\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Also, since the rod floats in any position, the sum of the moments of the forces about one end must vanish.

$$\therefore\ 4\times 2\times\rho+8\times 8\times{\cdot}24=x\times\frac{x}{2}\times 1\ \ldots\ldots\ldots\ldots(2).$$

Solving, we have $\rho=1{\cdot}32$, and $x=7{\cdot}2$.

12. Let V, V' be the volumes of the wood and lead, σ, σ' the sp. gr. of the wood and lead. Then

$$V\sigma w=6\,;\ V'(\sigma'-1)\,w=12\,;\ V(\sigma-1)\,w+V'(\sigma'-1)\,w=10.$$

The latter two give $$V(\sigma-1)\,w=-2.$$

$$\therefore\ \frac{\sigma-1}{\sigma}=\frac{-2}{6}=-\frac{1}{3}.\quad \therefore\ \sigma=\frac{3}{4}=\cdot75.$$

13. Since it floats half immersed, the sp. gr. of the first liquid $=2\rho$. The density of the mixture

$$=\frac{1}{2}(2\rho+1)=\rho+\frac{1}{2}.$$

$$\therefore\ \left(\rho+\frac{1}{2}\right)\times\frac{3}{4}=\rho\times 1.\quad \therefore\ \rho=1\cdot5.$$

14. The pressure on 1st day is that due to a head

$=76$ cm. of mercury $+3\cdot4$ of water

$=76\times13\cdot6+3\cdot4$ of water.

On the 2nd day the head similarly

$$=74\times13\cdot6+3\cdot4.$$

Let V be the volume to which with this pressure the 3 litres has increased. Then

$$\frac{V}{3}=\frac{76\times13\cdot6+3\cdot4}{74\times13\cdot6+3\cdot4}=\frac{76\times4+1}{74\times4+1}=\frac{305}{297}.$$

$$\therefore\ V=\frac{305}{99}=3\tfrac{8}{99}\text{ litres.}$$

$\therefore$ fraction of the original air that has bubbled out

$$=\frac{8}{99}\div3\tfrac{8}{99}=\frac{8}{305}.$$

15. Let the surface of the mercury inside the tube rise y inches, and that of the upper end of the water x inches. Now the area of the section of the bulb is 6^2, *i.e.* 36, times the area of the tube. Hence since the volume of the water is constant,

$$x=36y.$$

Also, if $\sigma=$ sp. gr. of the mercury,

$$\frac{1}{2}\sigma=\text{increase of pressure}=y\,.\,\sigma+(x-y)\,.\,1.$$

$$\therefore\ 18\sigma=x\sigma+35x,$$

so that $$x=\frac{18\sigma}{\sigma+35}$$

$$=\frac{18\times13\cdot67}{13\cdot67+35}=\text{etc.}$$

16. Required ratio

$$=\frac{\text{difference of wts. of air and gas in second case}}{\text{difference of wts. of air and gas in first case}}$$

$$=\frac{750-\frac{1}{10}.760}{760-\frac{1}{10}.760}=\frac{750-76}{760-76}=\frac{674}{684}=\frac{337}{342}.$$

17. When the sphere is at rest let Π', Π be the pressures of the air below and above the sphere. Then

$\pi r^2 kw=$ difference of pressures $=\pi r^2[\Pi'-\Pi]$, as in Art. 52,

so that $$kw=\Pi'-\Pi \text{ ....(1).}$$

Also, by Boyle's Law,

$$\Pi'\left[\pi r^2(h-x)-\frac{2}{3}\pi r^3\right]=\Pi\left[\pi r^2 h-\frac{2}{3}\pi r^3\right],$$

so that $$\Pi'\left[h-x-\frac{2r}{3}\right]=\Pi\left[h-\frac{2r}{3}\right] \text{ ....(2).}$$

(1) and (2) give, on eliminating Π',

$$kw\left[h-x-\frac{2r}{3}\right]=\Pi\,.\,x=wHx.$$

$$\therefore\ x(H+k)=k\left(h-\frac{2r}{3}\right).$$

18. Let ρ_1, σ_1 be the respective sp. gravities after the first mixing; ρ_2, σ_2 after the second, and so on. Then, by Art. 24,

$$\rho_1=\frac{n-1}{n}\rho+\frac{\sigma}{n} \text{ ....(1),}$$

and $$\sigma_1=\frac{\rho}{n}+\frac{n-1}{n}\sigma \text{ ....(2).}$$

$$\therefore\ \rho_1-\sigma_1=\left(1-\frac{2}{n}\right)(\rho-\sigma).$$

So $$\rho_2-\sigma_2=\left(1-\frac{2}{n}\right)(\rho_1-\sigma_1),$$

....

and $$\rho_m-\sigma_m=\left(1-\frac{2}{n}\right)(\rho_{m-1}-\sigma_{m-1}).$$

Therefore, by multiplication,

$$\rho_m-\sigma_m=\left(1-\frac{2}{n}\right)^m(\rho-\sigma) \text{ ....(3).}$$

Also clearly

$$\rho_m+\sigma_m=\rho_{m-1}+\sigma_{m-1}=\ldots=\rho+\sigma \quad \ldots\ldots\ldots\ldots\ldots(4).$$

Adding (3) and (4), we have

$$2\rho_m=\rho+\sigma+(\rho-\sigma)\left[1-\frac{2}{n}\right]^m,$$

i.e.

$$\rho_m=\rho+\frac{\sigma-\rho}{2}\left[1-\left(1-\frac{2}{n}\right)^m\right].$$

So for σ_m.

19. Let h_1 be the height of the required cylinder, and x the depth to which it is immersed, so that $x=\sigma h_1$.

If the volume of this depth x be just equal to one quarter of the outside cylinder, the water will just be on the point of overflowing. Hence

$$\pi r_1^2 . x=\frac{1}{4}\pi r^2 h.$$

$$\therefore\ h_1=\frac{x}{\sigma}=\frac{1}{4\sigma}.\frac{r^2}{r_1^2}.h.$$

20. Let O be the required point; since the depths of the $\angle^r$ points of the $\triangle^s$ OBA, OAC are the same, the depths of their C.G.'s are the same [*Statics*, Art. 104]. Hence if the pressures are to be the same their areas must be equal. Since $\triangle BOA=\triangle OAC$ it easily follows that O must lie on the median AD.

Also

$$\frac{1}{3}=\frac{\text{pressure on }\triangle BOC}{\text{pressure on }\triangle BAC}=\frac{\text{area of }BOC\times\text{depth of its C.G.}}{\text{area of }BAC\times\text{depth of its C.G.}}$$

$$=\frac{OD\times\text{depth of }O}{AD\times\text{depth of }A}=\frac{OD\times OD}{DA\times DA}.$$

$$\therefore\ \frac{1}{\sqrt{3}}=\frac{OD}{DA}.$$

21. Let ρ, σ be the densities of the cone and liquid. Then, since $\frac{3}{4}$ of the axis of the cone is immersed,

$$\therefore\ \sigma\times\left(\frac{3}{4}\right)^3=\rho . 1,$$

since the volumes of similar cones are as the cubes of homologous sides.

$$\therefore\ \rho:\sigma::27:64.$$

Also if a be the radius of the base of the cone, and hence that of the cylinder, the volume of the water which is above the vertex of the cone

$$=\pi a^2 \cdot \frac{3h}{4} - \frac{1}{3}\pi \cdot \frac{3h}{4}\left(\frac{3a}{4}\right)^2.$$

If the surface fall a distance y after the cone is removed, this must

$$=\pi a^2\left(\frac{3h}{4} - y\right).$$

Equating, we have

$$y = \frac{9h}{64}.$$

22. The thrust on the plane surface of the hemisphere

$$=\pi \cdot 3^2 \cdot 6w = 54\pi w. \qquad \text{[Art. 52.]}$$

The resultant thrust on the hemisphere is vertical and

$$= \text{the weight of the displaced fluid}$$

$$= \frac{2}{3}\pi \cdot 3^3 w = 18\pi w.$$

The required thrust is that which compounded with the first gives the second, and thus

$$=18\pi w\sqrt{3^2+1}$$

at an $\angle \tan^{-1}\frac{1}{3}$ to the horizon.

23. Let $ABCD$ be the square formed by the rods, A being lowest and AB being inclined at θ to the horizon. If possible let it float with the surface cutting AB at P and AD at Q; let $AB=a$, $AP=x$, so that $AQ=x\tan\theta$.

Let ρ = density of the rods; σ = that of the fluid. Then

$$(x + x\tan\theta)\,\sigma = 4a \cdot \rho \text{ ....(1).}$$

Also, taking moments about A, we have

$$\left(x \cdot \frac{x}{2}\cos\theta - x\tan\theta \cdot \frac{x\tan\theta}{2} \cdot \sin\theta\right)\sigma$$

$$= 4a\rho \times \frac{1}{2}a\sqrt{2}\cos(45^\circ + \theta) = 2a^2\rho(\cos\theta - \sin\theta),$$

so that $$x^2(1-\tan^3\theta)\,\sigma = 4a^2\rho(1-\tan\theta) \text{ ....(2).}$$

Squaring (1) and dividing by equation (2), we have on dividing by $(1-\tan\theta)$, and reducing,

$$\tan^2\theta(\sigma - 4\rho) + 2\tan\theta(\sigma - 2\rho) + \sigma - 4\rho = 0 \text{ ....(3).}$$

For this equation to have real roots we must have

$$(\sigma-2\rho)^2-(\sigma-4\rho)^2 \text{ positive,}$$

i.e. $$4\rho\sigma-12\rho^2 \text{ positive,}$$

i.e. $$\sigma>3\rho.$$

If $\sigma>4\rho$, all the terms of the equation (3) would be positive, and hence the value of $\tan\theta$ found would be negative, which is clearly impossible from the figure. If $\sigma=4\rho$, then (3) gives $\tan\theta=0$, and (1) gives $x=a$, which is the limiting case where one diagonal is horizontal.

$$\therefore\ \sigma>3\rho \text{ and } <4\rho.$$

24. The portion $\frac{W}{s}$ of the weight of W is borne by the water, and increases the tension on the other side.

$$\therefore\ W'+\frac{W}{s}=\text{tension of the supporting chain}=W-\frac{W}{s}.$$

$$\therefore\ s=\frac{2W}{W-W'}.$$

Again, the weight of the water alone $\not> W'$.

$$\therefore\ \text{volume of the water} \not> \frac{W'}{w}.$$

$$\therefore\ \frac{\text{volume of the water}}{\text{volume of the weight } W} \not> \frac{\frac{W'}{w}}{\frac{W}{sw}} \not> s\frac{W'}{W},$$

and therefore $$\not> \frac{2W'}{W-W'}.$$

25. Let σ_1, σ_2 be the spec. gravities of the water and cylinder; ρ and ρ' those of the air in the two cases; x and x' the portions of the cylinder immersed in the two cases; h its height. Then

$$h\sigma_1=x\sigma_1+(h-x)\,\rho,$$

and $$h\sigma_2=x'\sigma_1+(h-x')\,\rho'.$$

$$\therefore\ x-x'=h\frac{\sigma_2-\rho}{\sigma_1-\rho}-h\frac{\sigma_2-\rho'}{\sigma_1-\rho'}$$

$$=h\frac{(\sigma_1-\sigma_2)\,(\rho'-\rho)}{(\sigma_1-\rho)\,(\sigma_1-\rho')}=\text{positive, since } \rho'>\rho.$$

Hence $x>x'$, and the cylinder rises in the water.

Also the surface of the water inside is lower than that of the water outside by z, where $\Pi'-\Pi=wz$, *i.e.*

$$z=\frac{\Pi}{w}\left[\frac{\Pi'}{\Pi}-1\right]=H\,.\,\frac{\rho'-\rho}{\rho},$$

where H is the height of the water barometer originally.

Hence the cylinder goes *down* in space z and *up* through $x - x'$.

Hence it falls if $z > x - x'$, *i.e.* if

$$\frac{H}{\rho} > h \frac{\sigma_1 - \sigma_2}{(\sigma_1 - \rho)(\sigma_1 - \rho')}$$

and this will be so for any ordinary values of the symbols. Hence in general it falls.

26. If we assume AB to be inclined at an $\angle\,\theta$ to the horizon, and take moments about A for the two rods together, we easily have $\tan\theta = 1$, so that the only position of equilibrium is a symmetrical one.

Let ρ, σ be the densities of the rod and the water.

Resolving vertically, we have

$$a\rho = b\sigma \quad\ldots\ldots\ldots\ldots(1).$$

Taking moments about A for one rod, we have

$$2a\rho w \cdot \frac{a}{\sqrt{2}} = 2b\sigma w \cdot \frac{b}{\sqrt{2}} + T \cdot \frac{2a}{\sqrt{2}} \quad\ldots\ldots\ldots\ldots(2).$$

$$\therefore\ T = (a - b)\,\rho w = \frac{a - b}{2a}\,W.$$

27. The volumes of the portions of a cone which has its axis divided by planes parallel to the base into equal portions are in the ratio

$$h^3 : (2h)^3 - h^3 : (3h)^3 - (2h)^3,$$

i.e. as $1 : 7 : 19$.

Let ρ be the density of the cone, and $\sigma_1 - \sigma_2$, σ_1, and $\sigma_2 + \sigma_2$ those of the liquids.

Then $\qquad 27\rho = 19\,(\sigma_1 - \sigma_2) + 7 \cdot \sigma_1 + (\sigma_1 + \sigma_2),$

and $\qquad 27\rho = 7\,(\sigma_1 - \sigma_2) + 19\sigma_1.$

$$\therefore\ 27\rho = 27\sigma_1 - 18\sigma_2 = 26\sigma_1 - 7\sigma_2.$$

$$\therefore\ \sigma_1 = 11\sigma_2, \text{ and } \rho = \frac{31}{3}\sigma_2.$$

Hence $\qquad \sigma_1 - \sigma_2 = 10\sigma_2$, and $\sigma_1 + \sigma_2 = 12\sigma_2$. $\therefore$ etc.

28. Let the required depth be x; the stretched length of the string is now $a + x$, so that its tension $= nW\dfrac{x}{a}$, where W is the wt. of the cylinder. Hence, for equilibrium,

$W =$ tension of the string + wt. of the fluid displaced

$$= nW\frac{x}{a} + \frac{x}{h}\frac{W}{\sigma}.$$

$$\therefore\ x = \frac{ha\sigma}{nh\sigma + a}.$$

29. Let h cms. be the ht. of the water-barometer, so that the pressure inside the cylinder is that due to $(h+1700)$ cms.

Boyle's Law then gives

$$1200000 \times h = 450000 \times (h+1700).$$

$$\therefore \; h = 1020.$$

Also $\qquad p = w \times (1700 + 1020) = 2720 \times 980$ dynes.

30. Let the free surface, a parabola of lat.-rect. $\frac{2g}{\omega^2}$, meet the fixed leg in A and the revolving leg in P. Draw PN perpendicular to the fixed leg. Then $c^2 = \frac{2g}{\omega^2} \cdot AN$, giving $AN = \frac{\omega^2 c^2}{2g}$.

Let the heights of A and P above the bottoms of the legs be h, k. Then the mean level $= \frac{\sigma_1 h + \sigma_2 k}{\sigma_1 + \sigma_2}$. Hence answer

$$= k - \frac{\sigma_1 h + \sigma_2 k}{\sigma_1 + \sigma_2} = \frac{\sigma_1 (k-h)}{\sigma_1 + \sigma_2} = \frac{\sigma_1}{\sigma_1 + \sigma_2} \cdot \frac{\omega^2 c^2}{2g}.$$

31. If σ be the sp. gr. of the plank, we have

$$w + W = \frac{2}{3} \frac{W}{\sigma} \quad \text{..............................(1).}$$

Let the man walk to the end A of the plank, the section through the length of the plank being $ABCD$, B and C being under the water and D above. Draw AE horizontal to cut CD in E. Then $\triangle AED$ must be $\frac{1}{3}$ of $ABCD$, so that

$$DE = \frac{2b}{3}, \text{ and } \therefore \tan\theta = \frac{2b}{3a}, \text{ where } \theta = \angle DAE.$$

The wt. of the plank is W acting at the centre O of the section.

The buoyancy of the water is equal to wt. of $AECB$ of water acting upward at its C.G., *i.e.* to wt. of $ABCD$ acting upward at its C.G. and wt. of AED acting downward at its C.G. Therefore taking moments about A,

$$W\left(\frac{1}{\sigma} - 1\right)\left(\frac{a}{2}\cos\theta + \frac{b}{2}\sin\theta\right) = \text{moment of } \triangle AED \text{ about } A$$

$$= \text{sum moments of wts. each } \frac{1}{3}\frac{W}{\sigma} \text{ at each of pts. } A, E, D$$

$$= \frac{1}{3}\frac{W}{\sigma} \frac{a\cos\theta + (a\cos\theta + DE\sin\theta)}{3}.$$

$$\therefore \left(\frac{1}{\sigma} - 1\right)\left(\frac{a}{2} + \frac{b}{2}\tan\theta\right) = \frac{1}{9\sigma}\left[2a + \frac{2b}{3}\tan\theta\right].$$

$$\therefore \frac{9a^2}{2} + 3b^2 = \frac{1}{\sigma}\left[\frac{5a^2}{2} + \frac{23b^2}{9}\right].$$

Then (1) gives

$$\frac{w}{W}=\frac{2}{3}\cdot\frac{1}{\sigma}-1=\frac{6\,(9a^2+6b^2)}{45a^2+46b^2}-1=\frac{9a^2-10b^2}{45a^2+46b^2}.$$

This gives the limiting value of w.

32. Let h be the height and a the radius of the base of the cone; let b be the radius of the section of the cone by the water line and c the radius of the section of the cylinder, so that

$$b^2=\frac{6}{19}c^2 \quad\text{.......... ....(1).}$$

When $\frac{19}{8}\times\frac{1}{3}\pi b^2\,.\,\frac{b}{a}h$ of water is poured into the cylinder and the cone *held* at rest, the level of the water rises by x, where

$$\pi c^2\,.\,x-\frac{1}{3}\frac{\pi a^2 h}{h^3}\cdot\left[\left(\frac{b}{a}h+x\right)^3-\left(\frac{b}{a}h\right)^3\right]=\frac{19}{24}\pi\frac{b^3h}{a},$$

i.e.

$$\frac{19b^2x}{6}-\frac{a^2}{3h^2}\cdot\left[\left(\frac{bh}{a}+x\right)^3-\left(\frac{bh}{a}\right)^3\right]=\frac{19}{24}\frac{b^3h}{a}.$$

Putting

$$x=\frac{bh}{a}\,.\,y,$$

we easily have

$$(1+y)^3-1=\frac{19}{2}\left(y-\frac{1}{4}\right),$$

giving

$$y=\frac{1}{2},$$

and

$$\therefore\ x=\frac{bh}{2a}.$$

The additional buoyancy of the water now

$$=\text{wt. of }\frac{1}{3}\frac{\pi a^2h}{h^3}\left[\left(\frac{bh}{a}+x\right)^3-\left(\frac{bh}{a}\right)^3\right]$$

$$=\text{wt. of }\frac{19}{24}\frac{\pi b^3h}{a}$$

of water, and this is just balanced by the additional weight of the water poured into the cone.

The latter therefore need not be held, *i.e.* it is in equilibrium.

33. When lowered the air inside is at a pressure due to 51 ft. Hence if x be the required number of cubic feet, then, by Art. 114,

$$\frac{51\times 84}{1+\alpha\,.\,7}=\frac{34\times(x+84)}{1+\alpha\,.\,0}.$$

$$\therefore\ x+84=\frac{273}{280}\cdot\frac{51\times 84}{34}=122{\cdot}85.$$

34. As in Art. 124,

$$T = W - wAx,$$

and
$$x^2 + (a+h)\,x - hb = 0.$$

Eliminating x, so as to get T in terms of a, we have

$$(W-T)^2 + wA\,(a+h)\,(W-T) - w^2A^2hb = 0.$$

On putting $W = n\,.\,wA$, $T = wAY$, and X for a, we get

$$Y^2 - 2XY - (2n+h)\,Y + nX + n^2 + nh - hb = 0,$$

where the value of T is wA times the ordinate of this curve drawn perpendicular to AB. This curve is a hyperbola whose centre is the point $\left(-\frac{n+h}{2}, \frac{n}{2}\right)$ and whose asymptotes are parallel to the lines $Y=0$ and $Y=2X$.

35. Let the bottom of the bell be sunk a distance x. Then, by Boyle's Law,

$$\frac{2}{3}\pi a^3\,(x - c + H) = \left(\frac{2}{3}\pi a^3 + \pi a^2 c\right) \times H.$$

$$\therefore\; x = c\left[1 + \frac{3}{2}\frac{H}{a}\right] \text{......................(1)}.$$

Let V be the required volume, so that

$$\left(V + \frac{2}{3}\pi a^3 + \pi a^2 c\right) \times H = \left(\frac{2}{3}\pi a^3 + \pi a^2 c\right)(H + x).$$

$$\therefore\; \frac{V}{\frac{2}{3}\pi a^3 + \pi a^2 c} = \frac{x}{H} = \frac{c}{H} + \frac{3}{2}\frac{c}{a}. \quad \therefore \text{ etc.}$$

36. Let OC be the horizontal and CB the vertical leg.

Let P, Q be the ends of the mercury in OC, CB, when there is equilibrium, so that $OP = CQ = d$. The pressure at Q being Π, let A [vertically below O] be the vertex of the corresponding free surface which goes through P. Then, by Art. 166,

$$\Pi = \frac{1}{2}\omega^2\rho\,.\,l^2 - \rho g\,.\,AN,$$

and
$$O = \frac{1}{2}\omega^2\rho d^2 - \rho g AO,$$

where QN is perpendicular to AO.

$$\therefore\; \Pi = \frac{1}{2}\omega^2\rho\,(l^2 - d^2) - \rho g\,.\,ON.$$

$$\therefore \quad \rho g(h+d)=\frac{1}{2}\omega^2\rho(l^2-d^2),$$

i.e.

$$\omega^2=\frac{2g(h+d)}{l^2-d^2}.$$

This gives

$$d^2+\frac{2g}{\omega^2}d+\frac{2gh}{\omega^2}-l^2=0.$$

This equation has a positive root only if

$$\frac{2gh}{\omega^2}-l^2 \text{ is negative,}$$

i.e. if

$$\omega^2>\frac{2gh}{l^2}.$$

37. Let the liquid revolve round the tangent at O, the end of the horizontal radius. Let PCR be the vertical diameter, P being the highest point. Let the free surface through P cut the tube again in Q and have its vertex at A, a point vertically below O. Then

$$\angle QCR=\theta.$$

Draw PM, QN perpendicular to tangent at O.

Then

$$a^2=PM^2=\frac{2g}{\omega^2}.AM,$$

and

$$(a-a\sin\theta)^2=QN^2=\frac{2g}{\omega^2}.AN.$$

Therefore, by subtraction,

$$a^2-a^2(1-\sin\theta)^2=\frac{2g}{\omega^2}.MN.$$

$$\therefore \quad a^2[2\sin\theta-\sin^2\theta]=\frac{2ga}{\omega^2}(1+\cos\theta)=\frac{2ga}{\omega^2}.2\cos^2\frac{\theta}{2}.$$

$$\therefore \quad a\omega^2\left[\tan\frac{\theta}{2}-\sin^2\frac{\theta}{2}\right]=g.$$

38. Let A be the lowest point and C the centre of the vessel. Draw a parabola, latus rectum $\frac{2g}{\omega^2}$, with vertex at A and axis vertical, and let it meet the circle in P, P'. Let PP' meet AC produced in N. Let $PCN=\theta$. Then

$$r^2\sin^2\theta=\frac{2g}{\omega^2}.AN=\frac{2gr}{\omega^2}(1+\cos\theta).$$

$$\therefore \quad 1-\cos\theta=\frac{2g}{\omega^2 r}\quad\text{.........................(1).}$$

Also $$n^3 \times \frac{4}{3} \pi a^3$$

$=$ volume of the liquid

$=$ volume of spherical portion PAP'

$-$ volume of paraboloidal portion PAP'

$$= \frac{\pi}{3}(a + a\cos\theta)\,[3a^2 - (a^2 - a^2\cos\theta + a^2\cos^2\theta)]$$
$$- \frac{1}{2}\pi a^2 \sin^2\theta\,(a + a\cos\theta), \text{ by page ix.}$$

$$\therefore\ \frac{4}{3}n^3 = \frac{1}{3}(1+\cos\theta)(2+\cos\theta-\cos^2\theta) - \frac{1}{2}\sin^2\theta\,(1+\cos\theta)$$
$$= \frac{1}{6}(1+\cos\theta)^3.$$

$$\therefore\ 2n = 1 + \cos\theta = 2 - \frac{2g}{\omega^2 r}, \text{ by equation (1).}$$

$$\therefore\ \omega^2 = \frac{g}{r(1-n)}.$$

If ω exceed this value, the liquid breaks into two at A, and thus a hole there would not allow any liquid to escape.

39. Let V be the lowest point of the cone, N the highest point of the axis, and A the vertex of the free surface, so that

$$r^2 = \frac{2g}{\omega^2}.AN = l.AN.$$

Take any point Q on the surface of the conical portion. Draw QM perpendicular to the axis and let $QM = y$. By Art. 166,

$$\frac{p}{\rho} = \frac{\omega^2}{2}QM^2 + g.AM = g\left[\frac{y^2}{l} + AM\right].$$

Now $$AM = h + r\cot a - VM - AN,$$

where h = height of the cylinder.

$$\therefore\ AM = h + r\cot a - y\cot a - \frac{r^2}{l}.$$

$$\therefore\ \frac{p}{\rho}\frac{l}{g} = y^2 + l\cot a\,(r-y) + lh - r^2$$
$$= \left(y - \frac{l}{2}\cot a\right)^2 + lr\cot a + lh - r^2 - \frac{l^2\cot^2 a}{4}.$$

The least value of this is when

$$y = \frac{l}{2}\cot a.$$

This value of y will give a point on the cone if

$$\frac{l}{2}\cot\alpha < r,$$

i.e. if
$$l < 2r\tan\alpha.$$

Also these values of y and l give a positive value to p.

40. Let the plane of the paper through the vertex V and the axis of the cone cut the base in COD, O being the centre. Let the free surface cut the axis VO in A, and VC, VD in P and P'. Let PP' cut the axis in N. Let $VN=y$. Then

$$y^2\tan^2\alpha = \frac{2g}{\omega^2}\,.\,AN = \frac{3h}{4}\tan^2\alpha\,.\,AN,$$

so that
$$AN = \frac{4y^2}{3h}.$$

Then
$$\frac{1}{4}\times\frac{1}{3}\pi h^3\tan^2\alpha = \text{volume } VPAP'$$

$$= \frac{1}{3}\pi y^3\tan^2\alpha + \pi y^2\tan^2\alpha \times \frac{1}{2}\,\frac{4y^2}{3h}.$$

$$\therefore\ 8y^4 + 4hy^3 - h^4 = 0,$$

so that
$$y = \frac{h}{2},$$

and therefore
$$OA = h - \frac{h}{2} - AN = \frac{h}{2} - \frac{h}{3} = \frac{h}{6}.$$

Let vertical lines through C, D meet the parabola in Q and Q', and let QQ' meet the axis in M. Then

$$h^2\tan^2\alpha = \frac{2g}{\omega^2}\,.\,AM = \frac{3h}{4}\tan^2\alpha\,.\,AM,$$

and therefore
$$AM = \frac{4h}{3}$$

and
$$OM = \frac{4h}{3} + \frac{h}{6} = \frac{3h}{2}.$$

Therefore thrust on base

$$= \pi h^2\tan^2\alpha\,.\,OM\,.\,w - \pi h^2\tan^2\alpha\,.\,\frac{1}{2}AM\,.\,w$$

$$= \pi h^2\tan^2\alpha\,.\,w\left[\frac{3h}{2} - \frac{2h}{3}\right] = \frac{5}{6}\,.\,\pi h^3\tan^2\alpha\,.\,w.$$

Therefore thrust on base : wt. of the water :: $\frac{5}{6} : \frac{3}{4}\times\frac{1}{3}$ :: 10 : 3.

41. The free surface has its vertex at A, some point on the axis of revolution, and passes through P and Q the ends of the liquid. Since the tube is half-full, PQ passes through the centre C. Draw PM, QN perpendicular to the axis. Then

$$QN^2=\frac{2g}{\omega^2}.AN$$

and

$$PM^2=\frac{2g}{\omega^2}.AM.$$

$$\therefore\ QN^2-PM^2=\frac{2g}{\omega^2}.MN.$$

$$\therefore\ 2p\,(QN-PM)=\frac{2g}{\omega^2}.MN.$$

$$\therefore\ \tan\theta=\frac{QN-PM}{MN}=\frac{g}{p\omega^2}.$$

42. Let the hemisphere be that formed by the revolution of the semi-circle BDC about its axis OD. Through B and C draw the free parabolic surface, having its vertex at A, some point on OD.

Then $$a^2=OC^2=\frac{2g}{\omega^2}.OA.$$

Also the paraboloid BAC=half the hemisphere.

$$\therefore\ \frac{1}{2}OA\times\pi a^2=\frac{1}{3}\pi a^3.$$

$$\therefore\ \frac{1}{2}.\frac{\omega^2a^2}{2g}=\frac{1}{3}a.$$

$$\therefore\ \omega=\sqrt{\frac{4}{3}.\frac{g}{a}}.$$

43. Let the vertical plane through the vertex V and the axis VGO cut the cone in the generators VB, VA, the former being horizontal and in the surface of the fluid, and let

$$VG=\frac{3}{4}VO,$$

so that G is the centre of gravity through which the resultant vertical thrust

$$\frac{1}{3}\pi a^2hw\left(=\frac{1}{3}\pi l^3\sin^2\alpha\cos\alpha\,.\,w\right)\text{ passes.}$$

The centre of pressure of the plane face is a point X in OA such that

$$OX=\frac{a}{4}.\quad\text{(Page 61, Ex. 2.)}$$

The thrust on the plane face

$$=\pi a^2 . a \cos\alpha . w = \pi l^3 \sin^3\alpha \cos\alpha . w.$$

Let the perpendicular to the plane face through X meet the vertical through G in P. The required resultant thrust must then be a force R through P at an angle θ to the horizon, such that the resultant of it and $\pi l^3 \sin^3\alpha\cos\alpha . w$ may be

$$\frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha . w.$$

$$\therefore\ R\cos\theta = \pi l^3 \sin^3\alpha\cos\alpha . w . \cos\alpha,$$

$$R\sin\theta = \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha . w - \pi l^3 \sin^3\alpha\cos\alpha . w . \sin\alpha.$$

$$\therefore\ \tan\theta = \frac{1-3\sin^2\alpha}{3\sin\alpha\cos\alpha},$$

and $$R = \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha . w\sqrt{(1-3\sin^2\alpha)^2 + 9\sin^2\alpha\cos^2\alpha}$$

$$= \frac{1}{3}\pi l^3 \sin^2\alpha\cos\alpha . w\sqrt{1+3\sin^2\alpha}.$$

Let the force R meet VB in K, and draw GM perpendicular to VK. Then

$$VK = VG\cos\alpha + (MG+GP)\cot\theta$$

$$= \frac{3}{4}l\cos^2\alpha + \left[\frac{3}{4}l\cos\alpha\sin\alpha + \frac{OX}{\cos\alpha}\right]\cot\theta$$

$$= \frac{3}{4}l\cos^2\alpha + \left[\frac{3}{4}l\cos\alpha\sin\alpha + \frac{l}{4}\tan\alpha\right]\frac{3\sin\alpha\cos\alpha}{1-3\sin^2\alpha}$$

$$= \frac{3l}{4}\left[\cos^2\alpha + \frac{3\cos^2\alpha\sin^2\alpha}{1-3\sin^2\alpha} + \frac{\sin^2\alpha}{1-3\sin^2\alpha}\right]$$

$$= \frac{3l}{4} . \frac{1}{1-3\sin^2\alpha}.$$

44. Let A be the fixed point, C the centre of the plane base, B the point at which the hemisphere is loaded so that $\angle ACB = 90°$. Let the vertical through G the centre of gravity meet the plane base in N. We then have a downward force nW at B, an upward force $W\left(\frac{1}{\sigma}-1\right)$ at G. The remaining force, the action at A, must therefore be vertical also, and hence ANB a straight line. The required inclination is then $CGN=\theta$.

Taking moments about A, we have

$$AB . n = AN\left(\frac{1}{\sigma}-1\right).$$

$$\therefore\ AN = \frac{\sigma}{1-\sigma} n \,.\, AB = \lambda na\sqrt{2}.$$

$$\therefore\ CN^2 = CA^2 + AN^2 - 2CA \,.\, AN \cos 45° = a^2[1 - 2\lambda n + 2\lambda^2 n^2].$$

$$\therefore\ \tan\theta = \frac{CN}{CG} = \frac{CN}{\frac{3}{8}a} = \frac{8}{3}\sqrt{1 - 2\lambda n + 2\lambda^2 n^2}.$$

If $$\sigma = \frac{1}{1+n},$$

then $$\lambda = \frac{\sigma}{1-\sigma} = \frac{1}{n},$$

and $$\tan\theta = \frac{8}{3},$$

so that $$\cos\theta = \frac{3}{\sqrt{73}}.$$

45. Let the dividing plane through the axis VO cut the cone in the generators VC and VD. Let AB be the diameter of the base which is perpendicular to CD.

For equilibrium the upward pressure on the cone must not be $> W$, the weight of the cone, *i.e.*

$$\pi a^2 h \,.\, w - \frac{1}{3}\pi a^2 hw \text{ not } > W,$$

i.e. $$\frac{1}{3}\pi a^2 hw \text{ not } > \frac{1}{2}W,$$

i.e. $$\text{given ratio not } > \frac{1}{2}.$$

If on VO we take K so that

$$VK = \frac{3}{4}VO,$$

then through it, by Arts. 52 and 154, acts the resultant horizontal thrust

$$ahw \,.\, \frac{2h}{3},$$

so that the moment about V of this horizontal thrust

$$= \frac{2ah^2 w}{3} \times \frac{3h}{4} = \frac{1}{2}ah^3 w.$$

Draw vertical lines CC', BB', DD' to meet the horizontal plane through V in C', B', and D'. The moment about V of the vertical thrust on the half-cone $VBCD$ = moment of the wt. of half-cylinder

$C'B'D'CBD$ acting at its C.G. – moment of the wt. of half-cone $VCBD$ of liquid acting at its C.G.

$$=\frac{\pi a^2}{2}wh\times OG' - \frac{1}{6}\pi a^2 wh \cdot \frac{3}{4}\cdot OG' = \frac{3}{8}\pi a^2 wh \cdot OG',$$

where G' is on OB and is the centre of gravity of the semi-circle CBD and therefore

$$OG' = \frac{4a}{3\pi}.$$

Hence the moment of the vertical thrust

$$=\frac{a^3 wh}{2}.$$

Also since C.G. of the arc CBD is dist. $\frac{2a}{\pi}$ from O, the moment of the weight of the half-shell about V

$$=\frac{W}{2}\times\frac{2}{3}\cdot\frac{2a}{\pi}=\frac{2}{3}\frac{Wa}{\pi}.$$

Hence for equilibrium

$$\frac{1}{2}ah^3w+\frac{a^3wh}{2} \text{ not} > \frac{2}{3}\frac{Wa}{\pi}.$$

$$\therefore\ \frac{1}{3}\pi a^2hw\left[\frac{h^2}{a^2}+1\right] \text{ not} > \frac{4}{9}\cdot W,$$

i.e. $$\text{wt. of water}\times[1+\cot^2\alpha] \text{ not} > \frac{4}{9}W.$$

$$\therefore\ \frac{\text{wt. of water}}{W} \text{ not} > \frac{4}{9}\sin^2\alpha.$$

46. Let B be the point of the rim where the weight nW is placed. Then since it is the greatest weight, the water-line must just pass through B. Let α be the inclination of OB to the horizon where O is the centre. Taking moments about O we have, since the pressure at each point of the shell is towards O,

$$OB\cos\alpha\cdot nW = OG\sin\alpha\cdot W = \frac{1}{2}OB\sin\alpha\cdot W,$$

i.e. $$\tan\alpha = 2n.$$

Resolving vertically, we have $(n+1)W=$ wt. of the water included in the part of the sphere cut off by a plane distant $a\sin\alpha$ from the centre

$$=\frac{\pi}{3}[a-a\sin\alpha]^2[2a+a\sin\alpha]$$

$$=\frac{2\pi}{3}a^3w\times\frac{1}{2}(1-\sin\alpha)^2(2+\sin\alpha). \qquad \therefore \text{ etc.}$$

47. With the notation of Art. 123, on solving the equation for x, we have

$$x=\frac{-(a+h)+\sqrt{(a+h)^2+4hb}}{2}.$$

The pressure inside is now that due to a height $x+a+h$ of water. Thus the corresponding height of the mercury-barometer

$$=\frac{x+a+h}{\sigma}=\frac{a+h+\sqrt{(a+h)^2+4hb}}{2\sigma}.$$

If a block of wood of volume V and sp. gr. ρ be floated in from the outside, let the length of the air-column inside the bell be now y, and let V' be the volume of the portion of the wood not in the water. Then we now have $(Ay-V')$ of air at pressure $y+a+h$. Hence

$$(Ay-V')(y+a+h)=Ah \,.\, b,$$

i.e. $$\left(y-\frac{V'}{A}\right)(y+a+h)=hb \quad\text{.....................(1).}$$

But the equation for x is

$$x(x+a+h)=hb \quad\text{..........................(2).}$$

Thus y must be greater than x; for otherwise each factor on the left-hand side of (1) would be less than the corresponding one on the left-hand of (2).

Hence the height of the barometer is increased, since it measures a water-pressure $y+a+h$ instead of $x+a+h$.

If the block fall from a shelf within the bell and floats, so that V' of it is out of the water and z is the length now corresponding to x, we have

$$(Az-V')(z+a+h)=(Ax-V)(x+a+h).$$

It follows that z cannot be $>x$. For if it were, then

$$z+a+h>x+a+h,$$

and $$Az-V'>Ax-V,$$

since V' is necessarily $<V$.

Therefore z must be $<x$.

Therefore the barometer falls.

48. Let d be the depth of the dock, H the height of the water-barometer, b the height of the bell. Then

$$(d+H-b+h)\,h=b\,.\,H,$$

i.e. $$h(d+H)=bH \quad\text{.........................(1)}$$

on neglecting squares of small quantities.

If x be the length occupied by the air when the lowest point of the bell is at a depth y, then similarly

$$x(y+H)=bH=h(d+H) \qquad \text{.....................(2)}.$$

Let W be the weight, and V the volume of the iron of which the bell is made. Then

$$W=Aaw\sigma=V\rho w \qquad \text{..........................(3)}.$$

The tension of the chain now

$$=\text{wt. of bell} - \text{wt. of displaced fluid}$$

$$=Aaw\sigma-[Ax+V]w\sigma$$

$$=Aw\sigma\left[a\left(1-\frac{\sigma}{\rho}\right)-x\right]$$

$$=Aw\sigma\left[a\left(1-\frac{\sigma}{\rho}\right)-\frac{h(d+H)}{y+H}\right].$$

This tension just becomes zero, in which case the bell will rise by itself, when

$$(y+H)a\left(1-\frac{\sigma}{\rho}\right)=h(d+H),$$

and this value of y will be positive if

$$h(d+H)>Ha\left(1-\frac{\sigma}{\rho}\right),$$

i.e. if
$$\frac{d}{H}>\frac{a}{h}\left(1-\frac{\sigma}{\rho}\right)-1.$$

49. When the bell is lowered, let x be the length of it occupied by air. Then the pressure inside is that due to a height of water $(x+a+h)$, and thus the sp. gr. of the air

$$=\frac{x+a+h}{h}\sigma.$$

Hence, if V be the volume of the body, then

$$(V-cA).1+cA.\sigma=\text{its wt.}=[V-(c+\gamma\sigma)A]1+A(c+\gamma\sigma)\frac{x+a+h}{h}\sigma,$$

i.e.
$$(x+a)(c+\gamma\sigma)=h\gamma(1-\sigma) \qquad \text{......................(1)}.$$

Also a volume $A(b-c)$ of air at pressure h is now a volume $A[x-c-\gamma\sigma]$ at pressure $x+a+h$.

$$\therefore\ (b-c)h=(x-c-\gamma\sigma)(x+a+h) \qquad \text{..............(2)}.$$

From the accurate equations (1) and (2) we eliminate x.

If we treat σ as negligible, then (1) becomes

$$(x+a)\,c=\gamma h,$$

and substituting in (2) we have

$$(b-c)\,h=\left(\frac{\gamma h}{c}-a-c\right)\times h\left(\frac{\gamma}{c}+1\right).$$

$$\therefore\ (\gamma+c)\,(\gamma h-ac-c^2)=bc^2-c^3,$$

i.e.
$$\gamma^2 h+c\gamma\,(h-a-c)-c^2\,(a+b)=0.$$

50. Let BP be the tube, B being the lowest point, Q any point on the tube; draw PN, QM perpendicular to the vertical through B. Let $BQ=x$, $BP=l$. Let A be the vertex of the free surface corresponding to which the pressure at P would be Π. Then (Art. 166)

$$\Pi=\rho\left[\frac{1}{2}\,\omega^2 . PN^2-g\,.\,AN\right].$$

So
$$\begin{aligned} p &= \text{pressure at } Q \\ &= \rho\left[\frac{1}{2}\,\omega^2 QM^2-g\,.\,AM\right]. \end{aligned}$$

$$\therefore\ \frac{p-\Pi}{\rho}=\frac{1}{2}\,\omega^2\sin^2\alpha\,(x^2-l^2)-g\,(x-l)\cos\alpha$$

$$=\frac{\omega^2\sin^2\alpha}{2}\left[x-\frac{g\cos\alpha}{\omega^2\sin^2\alpha}\right]^2-\frac{\omega^2 l^2}{2}\sin^2\alpha+gl\cos\alpha-\frac{g^2\cos^2\alpha}{2\omega^2\sin^2\alpha}.$$

The least value of p is thus where

$$x=\frac{g\cos\alpha}{\omega^2\sin^2\alpha},$$

and this least value of p must not be negative; for otherwise a vacuum would be formed in the tube. Hence

$$\frac{\Pi}{\rho}-\frac{\omega^2 l^2\sin^2\alpha}{2}+gl\cos\alpha-\frac{g^2\cos^2\alpha}{\omega^2\sin^2\alpha}$$

must be positive.

$$\therefore\ \frac{\omega^2 l^2\sin^2\alpha}{2}-gl\cos\alpha<\frac{\Pi}{\rho}-\frac{g^2\cos^2\alpha}{2\omega^2\sin^2\alpha}.$$

$$\therefore\ \frac{\omega^2\sin^2\alpha}{2}\left[l-\frac{g\cos\alpha}{\omega^2\sin^2\alpha}\right]^2<\frac{\Pi}{\rho}.$$

$$\therefore\ l<\frac{g\cos\alpha}{\omega^2\sin^2\alpha}+\frac{\sqrt{2\Pi\rho}}{\rho\omega^2\sin^2\alpha}$$

$$<\frac{g\rho\cos\alpha+\sqrt{2\Pi\rho}}{\omega^2\rho\sin^2\alpha}.$$

51. Let a vertical section through the common axis cut the free surface of the liquid in the parabola $QPAP'Q'$; Q and Q' being the points in which it meets the vessel, P and P' the points in which it cuts the solid cylinder and A the vertex. Draw QMQ', PNP' perpendicular to the common axis. Then

$$R^2 = \frac{2g}{\omega^2} \cdot AM,$$

and

$$r^2 = \frac{2g}{\omega^2} \cdot AN,$$

so that

$$AM = \frac{\omega^2 R^2}{2g}$$

and

$$AN = \frac{\omega^2 r^2}{2g} \quad \text{.................................(1).}$$

If h be the height of the solid cylinder and t the height of A above its lowest point, the condition of equilibrium gives

$$\pi r^2 h\rho = \pi r^2 (t + AN) - \text{wt. of paraboloid } PAP' \text{ of water}$$

$$= \pi r^2 \left(t + \frac{1}{2} AN \right).$$

$$\therefore \ h\rho = t + \frac{\omega^2 r^2}{4g} \quad \text{...........................(2).}$$

Also, if y be the height of the lowest point of the solid cylinder from the bottom of the vessel, and $\pi R^2 b$ be the total volume of the water, then

$$\pi R^2 b = \pi R^2 y + \pi (R^2 - r^2)(t + AN)$$
$$+ \left[\pi R^2 . NM - \frac{1}{2} \pi R^2 . AM + \frac{1}{2} \pi r^2 . AN \right] \text{...(3).}$$

On substituting from (1) and (2) and reducing, this gives

$$R^2 b = R^2 y + h\rho (R^2 - r^2) + \frac{\omega^2}{4g} R^2 (R^2 - r^2) \quad \text{............(4).}$$

Similarly if y_0 be the value of y originally, *i.e.* when $\omega = 0$, we have

$$R^2 b = R^2 y_0 + h\rho (R^2 - r^2) \quad \text{.....................(5).}$$

Therefore from (4), (5) we have

$$y_0 - y = \frac{\omega^2}{4g} (R^2 - r^2).$$

52. Let $ABCD$ be the section of the embankment; AB the internal face being vertical and DC the external face being inclined at θ to the horizontal. [AD is the lowest line and BC the highest.]

Then the thrust of the water on AB must be less than the tangential action along AD, *i.e.*

$$AB \times \frac{AB}{2} w < \mu \times \text{wt. } ABCD < \mu w \cdot \rho \cdot na \cdot \left[a + \frac{na}{2} \cot \theta\right],$$

i.e.

$$\frac{n^2a^2}{2} < \mu n a^2 \left[1 + \frac{n}{2} \cot \theta\right] \cdot \rho,$$

i.e.

$$\cot \theta > \frac{1}{\mu\rho} - \frac{2}{n}.$$

Also the thrust of the water must not be able to turn the whole block about D.

Hence, taking moments about D, we have, since centre of pressure of BA is at $\frac{2}{3}$rds of the depth of AB,

$$\frac{1}{3} AB \times \frac{n^2a^2w}{2} < na^2 w\rho \left(na \cot \theta + \frac{a}{2}\right)$$
$$+ \frac{1}{2} na \cdot na \cot \theta \cdot \frac{2}{3} na \cot \theta \cdot \rho w.$$

$$\therefore \frac{w}{6} n^3a^3 < na^3 w\rho \left(n \cot \theta + \frac{1}{2}\right) + \frac{1}{3} n^3a^3 \cot^2 \theta \rho w,$$

i.e.

$$n < \frac{\rho}{n} (6n \cot \theta + 3) + 2n\rho \cot^2 \theta.$$

Therefore limiting value of $\cot \theta$ is given by

$$\cot^2 \theta + \frac{3}{n} \cot \theta + \frac{3}{2n^2} - \frac{1}{2\rho} = 0,$$

i.e. limiting value of $\cot \theta$

$$= -\frac{3}{2n} + \sqrt{\frac{3}{4n^2} + \frac{1}{2\rho}}.$$

Hence, since $\cot \theta$ must be greater than either of these values found, θ must be less than either of the angles stated.

53. With a foot, a lb., and a second as the fundamental units, we have

$$p = g\rho h = 32 \times \frac{13 \times 1000}{16} \times \frac{5}{2} \text{ poundals.}$$

With the new units let x be the required measure, so that

$$x\,[M']\,[L']^{-1}\,[T']^{-2}=\frac{32\times 13\times 1000\times 5}{2\times 16}\,[M]\,[L]^{-1}\,[T]^{-2}.$$

$$\therefore\ x=13\times 1000\times 5\times\frac{[M]}{[M']}\times\frac{[L']}{[L]}\times\frac{[T']^2}{[T]^2}$$

$$=13\times 1000\times 5\times\frac{1}{\frac{1}{16}}\times 3\times[60]^2$$

$$=13\times 10^3\times 5\times 16\times 3\times 36\times 10^2$$

$$=11232\times 10^6.$$

54. Divide the depth z into a very large number, n, of equal portions. Then the value of gravity at a depth $\frac{rz}{n}$ is $a+b\frac{rz}{n}$. For an element, of depth $\frac{z}{n}$, gravity may be considered constant, so that the pressure due to this element

$$=\rho\,.\left(a+b\frac{rz}{n}\right)\times\frac{z}{n}.$$

Hence at the depth z the pressure = sum of all these elements for values of r from 1 to n, when n is made large,

$$=\underset{n=\infty}{\Sigma}\ \rho z\,.\left[\frac{a}{n}+\frac{bz}{n^2}r\right]$$

$$=\underset{n=\infty}{\text{Lt}}\ \rho z\,.\left[\frac{na}{n}+\frac{bz}{n^2}\frac{n\,(n+1)}{2}\right]$$

$$=\rho z\,.\left[a+\frac{bz}{2}\right]=\rho\left[az+\frac{1}{2}bz^2\right].$$

55. The cylinder is floating freely with a depth x_0 immersed, where

$$x_0\rho a=l\sigma a,$$

i.e.

$$x_0=\frac{l\sigma}{\rho}.$$

Let P be the force required to keep the cylinder immersed a depth x, where

$$x=x_0+\frac{r}{n}\,(l-x_0),$$

so that

$$x\rho ag=P+l\sigma ag.$$

$$\therefore\ P=ag\,(x\rho-l\sigma)=a\rho g\,(x-x_0)=a\rho g\,.\frac{r}{n}\,(l-x_0).$$

In pushing the cylinder down through a further small distance

$$\frac{1}{n}(l-x_0),$$

the work done by P

$$=P\times\frac{1}{n}(l-x_0)=a\rho g\,.\,\frac{r}{n^2}(l-x_0)^2.$$

The total work required is the sum of all such elements of work for all integral values of r from 1 to n, when in the limit n is made indefinitely great. Therefore total work

$$=\sum_{n=\infty} a\rho g\,.\,(l-x_0)^2\frac{1+2+\ldots+n}{n^2}$$

$$=\sum_{n=\infty} a\rho g\,.\,(l-x_0)^2\times\frac{\frac{1}{2}n(n+1)}{n^2}$$

$$=\frac{1}{2}a\rho g\,(l-x_0)^2=\frac{1}{2}a\rho g l^2\,.\,\frac{(\rho-\sigma)^2}{\rho^2}$$

$$=\frac{1}{2}gal^2\,\frac{(\rho-\sigma)^2}{\rho}.$$

[This example may be done by a method similar to that of the next example, or may be deduced from it by putting $A=\infty$.]

56. In the original position let x_0 be the part immersed and ξ_0 the height of the bottom of the wood from the bottom of the vessel. Then if Ab be the total volume of the liquid, we have

$$x_0\,.\,\rho=\sigma l$$

and

$$Ab=A\xi_0+(A-a)\,x_0.$$

$$\therefore\ x_0=\frac{\sigma l}{\rho}$$

and

$$\xi_0=b-\frac{A-a}{A}\times\frac{\sigma l}{\rho}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Similarly if ξ_1 be the value of ξ_0 when the wood is just immersed, then

$$\xi_1=b-\frac{A-a}{A}l\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Now the work done = increase of the potential energy of the system.

Also potential energy in the first position

$$=g\rho\,.\,\xi_0 A\,.\,\frac{\xi_0}{2}+(A-a)\,x_0\left(\xi_0+\frac{1}{2}x_0\right)\rho g+al\sigma\left(\xi_0+\frac{l}{2}\right)g,$$

and that in the final position

$$=g\rho\,.\,\xi_1 A\,.\,\frac{\xi_1}{2}+(A-a)\,l\left(\xi_1+\frac{1}{2}l\right)\rho g+al\sigma\left(\xi_1+\frac{l}{2}\right)g.$$

Therefore work done = difference

$$=\frac{A\rho g}{2}(\xi_1^2-\xi_0^2)+(A-a)\,\rho g\left[l\xi_1-x_0\xi_0+\frac{1}{2}l^2-\frac{1}{2}x_0^2\right]+al\sigma(\xi_1-\xi_0)\,g.$$

Now, from (1) and (2),

$$\xi_1-\xi_0=-\frac{A-a}{A}\,l\left(1-\frac{\sigma}{\rho}\right),$$

$$l\xi_1-x_0\xi_0=\left(1-\frac{\sigma}{\rho}\right)\left[lb-l^2\frac{A-a}{A}\left(1+\frac{\sigma}{\rho}\right)\right],$$

and

$$\frac{1}{2}(l^2-x_0^2)=\frac{1}{2}l^2\left(1-\frac{\sigma^2}{\rho^2}\right).$$

Therefore work

$$\div\left[\frac{A-a}{A}\,l\left(1-\frac{\sigma}{\rho}\right)g\right]$$

$$=-\frac{A\rho}{2}(\xi_1+\xi_0)+A\rho\left[b-l\,\frac{A-a}{A}\left(1+\frac{\sigma}{\rho}\right)+\frac{1}{2}l\left(1+\frac{\sigma}{\rho}\right)\right]-al\sigma$$

$$=\frac{A\rho}{2}[2b-\xi_1-\xi_0]-l(A-a)(\rho+\sigma)+\frac{1}{2}lA(\rho+\sigma)-al\sigma$$

$$=\frac{1}{2}(A-a)\,l\,(\rho+\sigma)-l(A-a)(\rho+\sigma)+\frac{1}{2}Al(\rho+\tau)-al\sigma$$

$$=al\frac{\rho-\sigma}{2}.$$

Therefore work done

$$=\frac{1}{2}agl^2\left(1-\frac{a}{A}\right)\frac{(\rho-\sigma)^2}{\rho}.$$

57. Let the rod float vertically with $2b$ of its length in a liquid of density $n\rho$, and a weight W attached to its lower end. Then

$$2bgn\rho=2ag\rho+W,$$

i.e.

$$nb=a+\frac{W}{2g\rho}\quad\text{.............................(1)},$$

where $2a$ is the length, and ρ the density of the rod.

When the rod is inclined at a small angle to the vertical, we have an upward thrust $2bgn\rho$ acting at a distance b from the lower end, and a downward thrust $2agn\rho$ acting at a distance a from the lower end.

Hence, for stable equilibrium,

$$2bgn\rho \,.\, \frac{b}{2} > 2ag\rho \,.\, a.$$

$$\therefore \quad nb^2 > a^2,$$

i.e. $$nb > a\sqrt{n}.$$

Therefore from (1) $$a + \frac{W}{2g\rho} > a\sqrt{n}.$$

$$\therefore \; W > (\sqrt{n} - 1) \,.\, 2g\rho a, \qquad \therefore \text{ etc.}$$

58. Let θ be the very small angle through which the axis is turned, and h' the length of the axis then immersed, so that the weight of the fluid displaced is very approximately

$$\frac{h'^3}{h^3} \,.\, 2W,$$

where h is the length of the axis of the cone.

$$\therefore \; \frac{h'^3}{h^3} \,.\, 2W = W + w.$$

$$\therefore \; \frac{h'}{h} = \sqrt[3]{\frac{W+w}{2W}} = \frac{1}{\sqrt[3]{2}} \text{ approximately.}$$

Also as in Ex. 3, Page 233, the height of the metacentre above the vertex of the cone

$$= \frac{3}{4} h' + \frac{3a^2}{4h^2} h' = \frac{3}{2} h', \text{ since } a = h.$$

Hence, by taking moments about the vertex, we have

$$(W + w) \,.\, \frac{3h'}{2} \sin\theta = \frac{3}{4} h \sin\theta \,.\, W + w\,(a\cos\theta + h\sin\theta),$$

i.e. since θ is small, on neglecting the second-order product $w\theta$, we have

$$W \,.\, 2h'\theta = h \,.\, \theta \,.\, W + \frac{4}{3} wa,$$

i.e. $$\theta \left[2\frac{h}{\sqrt[3]{2}} - h \right] = \frac{4}{3} \frac{wa}{W},$$

i.e. $$\theta\,[\sqrt[3]{4} - 1] = \frac{4}{3} \frac{w}{W}.$$

59. As in Ex. 2, Page 233, we have the value of HM for a cylinder.

In equilibrium let a length x of the cylinder be immersed in the outside fluid, so that

$$x\rho = h'\sigma + nh\rho \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let M, M' be the metacentres for the outside and inside fluids, so that, if O be the lowest point of the axis, we have

$$OM = \frac{x}{2} + \frac{a^2}{4x},$$

and

$$OM' = \frac{h'}{2} + \frac{a^2}{4h'}.$$

Then in the disturbed position we have

a thrust upwards at $M = \pi a^2 . x\rho$,

a thrust downwards at $M' = \pi a^2 . h'\sigma$,

and a thrust downwards at $G = \pi a^2 . nh\rho$.

Therefore for stability

$$x\rho\left(\frac{x}{2} + \frac{a^2}{4x}\right) > h'\sigma\left(\frac{h'}{2} + \frac{a^2}{4h'}\right) + nh\rho . \frac{h}{2},$$

i.e.

$$2x^2\rho > (2h'^2 + a^2)\,\sigma + (2nh^2 - a^2)\,\rho.$$

Hence from (1)

$$2\,[h'\sigma + nh\rho]^2 > (2h'^2 + a^2)\,\rho\sigma + (2nh^2 - a^2)\,\rho^2.$$

60. In the position of equilibrium let a portion of the cone, length of axis x, be immersed in the outer liquid so that

$$\frac{1}{3}\pi x^3 \tan^2\alpha . \rho = \frac{1}{3}\pi h'^3 \tan^2\alpha . \sigma + \frac{1}{3}\pi\rho n h^3 \tan^2\alpha,$$

and therefore

$$x^3\rho = h'^3\sigma + h^3 n\rho \quad \text{..........................(1).}$$

As in Page 233, Ex. 3, we have the value of HM for a cone floating vertex downwards. Let M, M' be the metacentres for the outside and inside liquids so that, if V be the vertex of the cone,

$$VM = VH + HM = \frac{3}{4}x + \frac{3}{4}\tan^2\alpha x = \frac{3}{4}x\sec^2\alpha,$$

and

$$VM' = \frac{3}{4}h'\sec^2\alpha.$$

In the disturbed position we have

a thrust upwards at $M = \frac{1}{3}\pi\tan^2\alpha . x^3\rho$,

a thrust downwards at $M' = \frac{1}{3}\pi\tan^2\alpha . h'^3\sigma$,

and a thrust downwards at $G = \frac{1}{3}\pi\tan^2\alpha . nh^3\rho$.

Hence, for stability,

$$x^3\rho \times \frac{3}{4} x \sec^2 \alpha > h'^3\sigma \times \frac{3}{4} h' \sec^2 \alpha + nh^3\rho \times \frac{2h}{3},$$

i.e.

$$x^4\rho > h'^4\sigma + \frac{8}{9} nh^4\rho \cos^2 \alpha.$$

Substituting for x from (1), we have

$$(h'^3\sigma + h^3n\rho)^4 > \rho \left[h'^4\sigma + \frac{8}{9} nh^4\rho \cos^2 \alpha \right]^3.$$

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